28^{th} United States of America Mathematical Olympiad

Part I 9 a.m. – 12 noon

April 27, 1999

1. Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

   (a) every square that does not contain a checker shares a side with one that does;

   (b) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

   Prove that at least $(n^2 - 2)/3$ checkers have been placed on the board.

2. Let $ABCD$ be a cyclic quadrilateral. Prove that

$$|AB - CD| + |AD - BC| \geq 2|AC - BD|.$$

3. Let $p > 2$ be a prime and let a, b, c, d be integers not divisible by p, such that

$$\{ra/p\} + \{rb/p\} + \{rc/p\} + \{rd/p\} = 2$$

   for any integer r not divisible by p. Prove that at least two of the numbers $a+b$, $a+c$, $a+d$, $b+c$, $b+d$, $c+d$ are divisible by p. (Note: $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.)

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Part II 1 p.m. – 4 p.m.

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4. Let $a_1, a_2, \ldots, a_n$ $(n > 3)$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n \geq n \qquad \text{and} \qquad a_1^2 + a_2^2 + \cdots + a_n^2 \geq n^2.$$

   Prove that $\max(a_1, a_2, \ldots, a_n) \geq 2$.

5. The Y2K Game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

6. Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.