27^{th} United States of America Mathematical Olympiad

Part I   9 a.m. -12 noon

April 28, 1998

1. Suppose that the set $\{1, 2, \cdots, 1998\}$ has been partitioned into disjoint pairs $\{a_i, b_i\}$ $(1 \le i \le 999)$ so that for all i, $|a_i - b_i|$ equals 1 or 6. Prove that the sum
$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{999} - b_{999}|$$
ends in the digit 9.

2. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent AB to $\mathcal{C}_2$ $(B \in \mathcal{C}_2)$. Let C be the second point of intersection of AB and $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

3. Let $a_0, a_1, \cdots, a_n$ be numbers from the interval $(0, \pi/2)$ such that
$$\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) + \cdots + \tan(a_n - \frac{\pi}{4}) \ge n - 1.$$
Prove that
$$\tan a_0 \tan a_1 \cdots \tan a_n \ge n^{n+1}.$$

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Part II 1 p.m. - 4 p.m.

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4. A computer screen shows a 98×98 chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Find, with proof, the minimum number of mouse clicks needed to make the chessboard all one color.

5. Prove that for each $n \geq 2$, there is a set S of n integers such that $(a-b)^2$ divides ab for every distinct $a, b \in S$.

6. Let $n \geq 5$ be an integer. Find the largest integer k (as a function of n) such that there exists a convex n-gon $A_1 A_2 \dots A_n$ for which exactly k of the quadrilaterals $A_i A_{i+1} A_{i+2} A_{i+3}$ have an inscribed circle. (Here $A_{n+j} = A_j$.)