

26<sup>th</sup> United States of America Mathematical Olympiad

Part I 9 a.m. - 12 noon

May 1, 1997

1. Let  $p_1, p_2, p_3, \dots$  be the prime numbers listed in increasing order, and let  $x_0$  be a real number between 0 and 1. For positive integer  $k$ , define

$$x_k = \begin{cases} 0 & \text{if } x_{k-1} = 0, \\ \left\{ \frac{p_k}{x_{k-1}} \right\} & \text{if } x_{k-1} \neq 0, \end{cases}$$

where  $\{x\}$  denotes the fractional part of  $x$ . (The fractional part of  $x$  is given by  $x - [x]$  where  $[x]$  is the greatest integer less than or equal to  $x$ .) Find, with proof, all  $x_0$  satisfying  $0 < x_0 < 1$  for which the sequence  $x_0, x_1, x_2, \dots$  eventually becomes 0.

2. Let  $ABC$  be a triangle, and draw isosceles triangles  $BCD, CAE, ABF$  externally to  $ABC$ , with  $BC, CA, AB$  as their respective bases. Prove that the lines through  $A, B, C$  perpendicular to the lines  $\overleftrightarrow{EF}, \overleftrightarrow{FD}, \overleftrightarrow{DE}$ , respectively, are concurrent.
3. Prove that for any integer  $n$ , there exists a unique polynomial  $Q$  with coefficients in  $\{0, 1, \dots, 9\}$  such that  $Q(-2) = Q(-5) = n$ .

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Part II 1 p.m. - 4 p.m.

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4. To *clip* a convex  $n$ -gon means to choose a pair of consecutive sides  $AB, BC$  and to replace them by the three segments  $AM, MN$ , and  $NC$ , where  $M$  is the midpoint of  $AB$  and  $N$  is the midpoint of  $BC$ . In other words, one cuts off the triangle  $MBN$  to obtain a convex  $(n + 1)$ -gon. A regular hexagon  $\mathcal{P}_6$  of area 1 is clipped to obtain a heptagon  $\mathcal{P}_7$ . Then  $\mathcal{P}_7$  is clipped (in one of the seven possible ways) to obtain an octagon  $\mathcal{P}_8$ , and so on. Prove that no matter how the clippings are done, the area of  $\mathcal{P}_n$  is greater than  $1/3$ , for all  $n \geq 6$ .
5. Prove that, for all positive real numbers  $a, b, c$ ,

$$(a^3 + b^3 + abc)^{-1} + (b^3 + c^3 + abc)^{-1} + (c^3 + a^3 + abc)^{-1} \leq (abc)^{-1}.$$

6. Suppose the sequence of nonnegative integers  $a_1, a_2, \dots, a_{1997}$  satisfies

$$a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1$$

for all  $i, j \geq 1$  with  $i + j \leq 1997$ . Show that there exists a real number  $x$  such that  $a_n = \lfloor nx \rfloor$  (the greatest integer  $\leq nx$ ) for all  $1 \leq n \leq 1997$ .