

26th United States of America Mathematical Olympiad

Part I 9 a.m. - 12 noon

May 1, 1997

1. Let p_1, p_2, p_3, \dots be the prime numbers listed in increasing order, and let x_0 be a real number between 0 and 1. For positive integer k , define

$$x_k = \begin{cases} 0 & \text{if } x_{k-1} = 0, \\ \left\{ \frac{p_k}{x_{k-1}} \right\} & \text{if } x_{k-1} \neq 0, \end{cases}$$

where $\{x\}$ denotes the fractional part of x . (The fractional part of x is given by $x - [x]$ where $[x]$ is the greatest integer less than or equal to x .) Find, with proof, all x_0 satisfying $0 < x_0 < 1$ for which the sequence x_0, x_1, x_2, \dots eventually becomes 0.

2. Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC , with BC, CA, AB as their respective bases. Prove that the lines through A, B, C perpendicular to the lines $\overleftrightarrow{EF}, \overleftrightarrow{FD}, \overleftrightarrow{DE}$, respectively, are concurrent.
3. Prove that for any integer n , there exists a unique polynomial Q with coefficients in $\{0, 1, \dots, 9\}$ such that $Q(-2) = Q(-5) = n$.

26th United States of America Mathematical Olympiad

Part II 1 p.m. - 4 p.m.

May 1, 1997

4. To *clip* a convex n -gon means to choose a pair of consecutive sides AB, BC and to replace them by the three segments AM, MN , and NC , where M is the midpoint of AB and N is the midpoint of BC . In other words, one cuts off the triangle MBN to obtain a convex $(n + 1)$ -gon. A regular hexagon \mathcal{P}_6 of area 1 is clipped to obtain a heptagon \mathcal{P}_7 . Then \mathcal{P}_7 is clipped (in one of the seven possible ways) to obtain an octagon \mathcal{P}_8 , and so on. Prove that no matter how the clippings are done, the area of \mathcal{P}_n is greater than $1/3$, for all $n \geq 6$.
5. Prove that, for all positive real numbers a, b, c ,

$$(a^3 + b^3 + abc)^{-1} + (b^3 + c^3 + abc)^{-1} + (c^3 + a^3 + abc)^{-1} \leq (abc)^{-1}.$$

6. Suppose the sequence of nonnegative integers $a_1, a_2, \dots, a_{1997}$ satisfies

$$a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1$$

for all $i, j \geq 1$ with $i + j \leq 1997$. Show that there exists a real number x such that $a_n = \lfloor nx \rfloor$ (the greatest integer $\leq nx$) for all $1 \leq n \leq 1997$.