

25th United States of America Mathematical Olympiad

Part I 9 a.m. - 12 noon

May 2, 1996

1. Prove that the average of the numbers $n \sin n^\circ$ ($n = 2, 4, 6, \dots, 180$) is $\cot 1^\circ$.
2. For any nonempty set S of real numbers, let $\sigma(S)$ denote the sum of the elements of S . Given a set A of n positive integers, consider the collection of all distinct sums $\sigma(S)$ as S ranges over the nonempty subsets of A . Prove that this collection of sums can be partitioned into n classes so that in each class, the ratio of the largest sum to the smallest sum does not exceed 2.
3. Let ABC be a triangle. Prove that there is a line ℓ (in the plane of triangle ABC) such that the intersection of the interior of triangle ABC and the interior of its reflection $A'B'C'$ in ℓ has area more than $2/3$ the area of triangle ABC .

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Part II 1 p.m. - 4 p.m.

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4. An n -term sequence (x_1, x_2, \dots, x_n) in which each term is either 0 or 1 is called a *binary sequence of length n* . Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n .
5. Triangle ABC has the following property: there is an interior point P such that $\angle PAB = 10^\circ$, $\angle PBA = 20^\circ$, $\angle PCA = 30^\circ$, and $\angle PAC = 40^\circ$. Prove that triangle ABC is isosceles.
6. Determine (with proof) whether there is a subset X of the integers with the following property: for any integer n there is exactly one solution of $a + 2b = n$ with $a, b \in X$.