

23rd United States of America Mathematical Olympiad

April 28, 1994

Time Limit: 3½ hours

1. Let $k_1 < k_2 < k_3 < \dots$ be positive integers, no two consecutive, and let $s_m = k_1 + k_2 + \dots + k_m$ for $m = 1, 2, 3, \dots$. Prove that, for each positive integer n , the interval $[s_n, s_{n+1})$ contains at least one perfect square.
2. The sides of a 99-gon are initially colored so that consecutive sides are red, blue, red, blue, \dots , red, blue, yellow. We make a sequence of modifications in the coloring, changing the color of one side at a time to one of the three given colors (red, blue, yellow), under the constraint that no two adjacent sides may be the same color. By making a sequence of such modifications, is it possible to arrive at the coloring in which consecutive sides are red, blue, red, blue, red, blue, \dots , red, yellow, blue?
3. A convex hexagon $ABCDEF$ is inscribed in a circle such that $AB = CD = EF$ and diagonals AD , BE , and CF are concurrent. Let P be the intersection of AD and CE . Prove that $CP/PE = (AC/CE)^2$.
4. Let a_1, a_2, a_3, \dots be a sequence of positive real numbers satisfying $\sum_{j=1}^n a_j \geq \sqrt{n}$ for all $n \geq 1$. Prove that, for all $n \geq 1$,

$$\sum_{j=1}^n a_j^2 > \frac{1}{4} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} \right).$$

5. Let $|U|$, $\sigma(U)$ and $\pi(U)$ denote the number of elements, the sum, and the product, respectively, of a finite set U of positive integers. (If U is the empty set, $|U| = 0$, $\sigma(U) = 0$, $\pi(U) = 1$.) Let S be a finite set of positive integers. As usual, let $\binom{n}{k}$ denote $\frac{n!}{k!(n-k)!}$. Prove that

$$\sum_{U \subseteq S} (-1)^{|U|} \binom{m - \sigma(U)}{|S|} = \pi(S)$$

for all integers $m \geq \sigma(S)$.