21^{st} USA Mathematical Olympiad

April 30, 1992
Time Limit: $3\frac{1}{2}$ hours

1. Find, as a function of n, the sum of the digits of
$$9 \times 99 \times 9999 \times \cdots \times \left(10^{2^n} - 1\right),$$
where each factor has twice as many digits as the previous one.

2. Prove
$$\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{1}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}.$$

3. For a nonempty set S of integers, let $\sigma(S)$ be the sum of the elements of S. Suppose that $A = \{a_1, a_2, \ldots, a_{11}\}$ is a set of positive integers with $a_1 < a_2 < \cdots < a_{11}$ and that, for each positive integer $n \le 1500$, there is a subset S of A for which $\sigma(S) = n$. What is the smallest possible value of a_{10}?

4. Chords $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ of a sphere meet at an interior point P but are not contained in a plane. The sphere through A, B, C, P is tangent to the sphere through A', B', C', P. Prove that $AA' = BB' = CC'$.

5. Let $P(z)$ be a polynomial with complex coefficients which is of degree 1992 and has distinct zeros. Prove that there exist complex numbers $a_1, a_2, \ldots, a_{1992}$ such that $P(z)$ divides the polynomial
$$\left(\cdots\left((z-a_1)^2 - a_2\right)^2 \cdots - a_{1991}\right)^2 - a_{1992}.$$