19^{th} USA Mathematical Olympiad

April 24, 1990
Time Limit: $3\frac{1}{2}$ hours

1. A certain state issues license plates consisting of six digits (from 0 through 9). The state requires that any two plates differ in at least two places. (Thus the plates $\boxed{027592}$ and $\boxed{020592}$ cannot both be used.) Determine, with proof, the maximum number of distinct license plates that the state can use.

2. A sequence of functions $\{f_n(x)\}$ is defined recursively as follows:
$$\begin{aligned} f_1(x) &= \sqrt{x^2+48}, \quad \text{and} \\ f_{n+1}(x) &= \sqrt{x^2+6f_n(x)} \quad \text{for } n \geq 1. \end{aligned}$$
(Recall that $\sqrt{}$ is understood to represent the positive square root.) For each positive integer n, find all real solutions of the equation $f_n(x) = 2x$.

3. Suppose that necklace A has 14 beads and necklace B has 19. Prove that for any odd integer $n \geq 1$, there is a way to number each of the 33 beads with an integer from the sequence
$$\{n, n+1, n+2, \ldots, n+32\}$$
so that each integer is used once, and adjacent beads correspond to relatively prime integers. (Here a "necklace" is viewed as a circle in which each bead is adjacent to two other beads.)

4. Find, with proof, the number of positive integers whose base-n representation consists of distinct digits with the property that, except for the leftmost digit, every digit differs by ± 1 from some digit further to the left. (Your answer should be an explicit function of n in simplest form.)

5. An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.