18^{th} USA Mathematical Olympiad

April 25, 1989
Time Limit: $3\frac{1}{2}$ hours

1. For each positive integer n, let

$$\begin{aligned} S_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \\ T_n &= S_1 + S_2 + S_3 + \cdots + S_n, \\ U_n &= \frac{T_1}{2} + \frac{T_2}{3} + \frac{T_3}{4} + \cdots + \frac{T_n}{n+1}. \end{aligned}$$

   Find, with proof, integers $0 < a, b, c, d < 1000000$ such that $T_{1988} = aS_{1989} - b$ and $U_{1988} = cS_{1989} - d$.

2. The 20 members of a local tennis club have scheduled exactly 14 two-person games among themselves, with each member playing in at least one game. Prove that within this schedule there must be a set of 6 games with 12 distinct players.

3. Let $P(z) = z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n$ be a polynomial in the complex variable z, with real coefficients c_k. Suppose that $|P(i)| < 1$. Prove that there exist real numbers a and b such that $P(a + bi) = 0$ and $(a^2 + b^2 + 1)^2 < 4b^2 + 1$.

4. Let ABC be an acute-angled triangle whose side lengths satisfy the inequalities $AB < AC < BC$. If point I is the center of the inscribed circle of triangle ABC and point O is the center of the circumscribed circle, prove that line IO intersects segments AB and BC.

5. Let u and v be real numbers such that

$$(u + u^2 + u^3 + \cdots + u^8) + 10u^9 = (v + v^2 + v^3 + \cdots + v^{10}) + 10v^{11} = 8.$$

   Determine, with proof, which of the two numbers, u or v, is larger.