Inequalities

1. Let $a, b, c \in \mathbf{R}_+$ so that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Prove that:

$$\sum \frac{1}{a-1} \geq \frac{3}{2}.$$

2. Let $a, b, c \in \mathbf{R}_+$ so that $a + b + c = 1$. Prove that:

$$\sum \frac{1}{1-a} \geq \frac{9}{2}.$$

3. Prove the following inequality for $a \in (1, \infty) \cap \mathbf{Q}$ and $n \in \mathbf{N^*}$:

$$\sum_{k=2}^{n+1} k^{2a} > a \cdot n(n+1).$$

4. Prove that for every $a, b, c \in \mathbf{R}_+$ satisfying $abc = 1$ and for every $n \in \mathbf{N^*}$ the following inequality holds:

$$\sum \frac{1}{a^{2^n+1}(b+c)} \geq \frac{3}{2}.$$

5. Let $a, b, c \in \mathbf{R}_+$ so that $a + b + c = 1$. Prove that:

$$\prod \frac{1+a}{1-a} \geq 8.$$

6. Let $a, b, c \in \mathbf{R}_+$ so that $a + b + c = 1$. Prove that:

$$\sum \frac{a^3}{1+b-c} \geq \frac{1}{9}.$$

7. Let $a, b, c \in \mathbf{R}_+$ so that $ab + bc + ca = 1$. Prove that:

$$\sum \frac{1-a^2}{1+a^2} \leq \frac{3}{2}.$$

8. Let $a, b, c \in \mathbf{R}_+$ so that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Prove that:

$$\sum (a-1)(b-1) \geq 12.$$

9. Let $a, b, c \in \mathbf{R}_+$ so that $a + b + c = abc$. Knowing that the nominators are positive prove that:

$$\sum \frac{a^7}{a^2 + 5bc - a^2bc} \geq 9\sqrt{3}.$$

10. Let $a, b, c \in \mathbf{R}_+$. Prove that:

$$\sum \frac{a^3}{(b+c)(a+1)} \geq \frac{27}{8}.$$

11. Let $a, b, c \in \mathbf{R}_+$ so that $a+b+c = abc$. Prove that:

$$\sum a \geq 3 \sum \frac{1}{a}.$$

12. Let $a, b, c \in \mathbf{R}_+$ so that $a+b+c = abc$. Prove that:

$$\sum \frac{a^2}{bc(ab+3)} \geq \frac{1}{2}.$$

13. Let $a, b, c \in \mathbf{R}_+$ so that $a+b+c = 1$. Prove that:

$$\sum \frac{a}{1-a^2} \geq \frac{27}{8}.$$

14. Let $a, b, c \in \mathbf{R}_+$ so that $a+b+c = abc$. Prove that:

$$\prod (a^2+1) \geq 8 \prod (bc-1).$$

15. Let $a, b, c \in \mathbf{R}_+$ so that $a+b+c = abc$. Prove that:

$$\prod \left(a - \frac{1}{b}\right) \geq \frac{8}{\sum a}.$$

16. Let $a, b, c \in \mathbf{R}_+$ so that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Prove that:

$$\prod (a-2) \leq 1.$$

17. Let $a, b, c \in \mathbf{R}_+$ so that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Prove that:

$$\prod (a-1) \geq 8 \prod (a-2).$$

18. Let $a, b, c \in \mathbf{R}_+$ so that $ab+bc+ca = 1$. Prove that:

$$\sum \left(\frac{bc}{a}\right)^2 \geq 1.$$

19. Let $a, b, c \in \mathbf{R}_+$ so that $ab+bc+ca = 1$. Prove that:

$$\sum \left(\frac{b+c}{\frac{1}{bc} - 1}\right)^2 \geq 1.$$

20. Let $a, b, c \in \mathbf{R}_+$ so that $a^2 + b^2 + c^2 = 3$. Prove that:

$$\sum \frac{1}{\sqrt{-a+b+c}} \geq \sum \sqrt{\frac{a}{2-a}}.$$

21. Let $a, b, c \in \mathbf{R}_+$ so that $a + b + c = 1$. Prove that:

$$\prod(a+b) \leq \frac{5}{4}abc + \frac{1}{4}.$$

22. Let $a, b, c \in \mathbf{R}_+$ so that $a + b + c = 1$. Prove that:

$$4(a^3 + b^3 + c^3) + 15abc \geq 1.$$

23. Let $a, b, c \in \mathbf{R}_+$ so that $a + b + c = 1$. Prove that:

$$\sum \frac{1}{a^2 + b^2 + 2c} \leq \frac{3\sqrt{3}}{8} \frac{1}{\sqrt{abc}}.$$

24. Let $a, b, c \in \mathbf{R}_+$ so that $a + b + c = 1$. Prove that:

$$\prod(a+b) \leq a^3 + b^3 + c^3 + 5abc.$$

25. Let $a, b, c \in \mathbf{R}_+$ so that $a + b + c = 1$. Prove that:

$$\sum \frac{1}{1-a} \leq \frac{1}{6abc}.$$

26. Let $a, b, c \in \mathbf{R}_+$ so that $a + b + c = 1$. Prove that:

$$\sum \frac{1}{\sqrt{a^3 + b^3}} \leq \frac{1}{\sqrt{6}abc}.$$

27. Let $x_1, x_2, ..., x_n > 0$ so that $x_1 + x_2 + ... + x_n = 1$. Prove that:

$$\sum_{k=1}^{n} x_k^2 + n^2 \leq \sum_{k=1}^{n} \frac{1}{x_k} + \frac{1}{n}.$$

All problems are proposed by **Călin Popa**.