

The 43rd International Mathematical Olympiad

1. Let n be a positive integer. Let T be the set of points (x, y) in the plane where x and y are non-negative integers and $x + y < n$. Each point of T is colored red or blue. If a point (x, y) is red, then so are all points (x', y') of T with both x' and y' . Define an X -set to be a set of n blue points having distinct x -coordinates, and a Y -set to be a set of n blue points having distinct y -coordinates. Prove that the number of X -sets is equal to the number of Y -sets.
2. Let BC be a diameter of the circle Γ with center O . Let A be a point on Γ such that $0^\circ < \angle AOB < 120^\circ$. Let D be the midpoint of the arc AB not containing C . The line through O parallel to DA meets the line AC at J . The perpendicular bisector of OA meets Γ at E and F . Prove that J is the incenter of the triangle $\triangle CEF$.

3. Find all pairs of integers $m, n \geq 3$ such that there exist infinitely many positive integers a for which

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is an integer.

4. Let n be an integer greater than 1. The positive divisors of n are d_1, d_2, \dots, d_k where $1 = d_1 < d_2 < \dots < d_k = n$. Define $D = d_1d_2 + d_2d_3 + \dots + d_{k-1}d_k$.
 - (a) Prove that $D < n^2$.
 - (b) Determine all n for which D is a divisor of n^2 .
5. Find all functions $f : \mathbf{R} \rightarrow \mathbf{R}$ such that:

$$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz), \forall x, y, z, t \in \mathbf{R}.$$

6. Let $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ be circles of radius 1 in the plane, where $n \geq 3$. Denote their centers by O_1, O_2, \dots, O_n respectively. Suppose that no line meets more than two of the circles. Prove that:

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_i O_j} \leq \frac{(n-1)\pi}{4}.$$

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