40^{th} International Mathematical Olympiad

Bucharest, Romania

Day I 9 a.m. - 1:30 p.m.

July 16, 1999

1. Find all finite sets S of at least three points in the plane such that for all distinct points A, B in S, the perpendicular bisector of AB is an axis of symmetry for S.

2. Let $n \geq 2$ be a fixed integer.

   (a) Find the least constant C such that for all nonnegative real numbers $x_1, \ldots, x_n$,

   $$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{i=1}^{n} x_i \right)^4 .$$

   (b) Determine when equality occurs for this value of C.

3. We are given an $n \times n$ square board, with n even. Two distinct squares of the board are said to be adjacent if they share a common side. (A square is *not* adjacent to itself.) Find the minimum number of squares that can be marked so that every square (marked or not) is adjacent to at least one marked square.

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4. Find all pairs (n, p) of positive integers such that
   - p is prime;
   - $n \leq 2p$;
   - $(p-1)^n + 1$ is divisible by n^{p-1}.

5. The circles Γ_1 and Γ_2 lie inside circle Γ, and are tangent to it at M and N, respectively. It is given that Γ_1 passes through the center of Γ_2. The common chord of Γ_1 and Γ_2, when extended, meets Γ at A and B. The lines MA and MB meet Γ_1 again at C and D. Prove that the line CD is tangent to Γ_2.

6. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1.$$
   for all $x, y \in \mathbb{R}$.