37^{th} International Mathematical Olympiad

Mumbai, India

Day I 9 a.m. - 1:30 p.m.

July 10, 1996

1. We are given a positive integer r and a rectangular board $ABCD$ with dimensions $|AB| = 20, |BC| = 12$. The rectangle is divided into a grid of 20×12 unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from the square with A as a vertex to the square with B as a vertex.

   (a) Show that the task cannot be done if r is divisible by 2 or 3.

   (b) Prove that the task is possible when $r = 73$.

   (c) Can the task be done when $r = 97$?

2. Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

   Let D, E be the incenters of triangles APB, APC, respectively. Show that AP, BD, CE meet at a point.

3. Let S denote the set of nonnegative integers. Find all functions f from S to itself such that

$$f(m + f(n)) = f(f(m)) + f(n) \qquad \forall m, n \in S.$$

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4. The positive integers a and b are such that the numbers $15a+16b$ and $16a-15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

5. Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to FA. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF, respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$

6. Let p, q, n be three positive integers with $p + q < n$. Let $(x_0, x_1, \ldots, x_n)$ be an $(n+1)$-tuple of integers satisfying the following conditions:

   (a) $x_0 = x_n = 0$.

   (b) For each i with $1 \leq i \leq n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.

   Show that there exist indices $i < j$ with $(i, j) \neq (0, n)$, such that $x_i = x_j$.