

33rd International Mathematical Olympiad

First Day - Moscow - July 15, 1992

Time Limit: 4½ hours

1. Find all integers a, b, c with $1 < a < b < c$ such that

$$(a - 1)(b - 1)(c - 1) \quad \text{is a divisor of } abc - 1.$$

2. Let \mathbf{R} denote the set of all real numbers. Find all functions $f : \mathbf{R} \rightarrow \mathbf{R}$ such that

$$f(x^2 + f(y)) = y + (f(x))^2 \quad \text{for all } x, y \in \mathbf{R}.$$

3. Consider nine points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either colored blue or red or left uncolored. Find the smallest value of n such that whenever exactly n edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.

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4. In the plane let C be a circle, L a line tangent to the circle C , and M a point on L . Find the locus of all points P with the following property: there exists two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR .
5. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz -plane, zx -plane, xy -plane, respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|,$$

where $|A|$ denotes the number of elements in the finite set A . (Note: The orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.)

6. For each positive integer n , $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive squares.
- (a) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.
- (b) Find an integer n such that $S(n) = n^2 - 14$.
- (c) Prove that there are infinitely many integers n such that $S(n) = n^2 - 14$.