

31<sup>st</sup> International Mathematical Olympiad

Beijing, China

Day I

July 12, 1990

1. Chords  $AB$  and  $CD$  of a circle intersect at a point  $E$  inside the circle. Let  $M$  be an interior point of the segment  $EB$ . The tangent line at  $E$  to the circle through  $D$ ,  $E$ , and  $M$  intersects the lines  $BC$  and  $AC$  at  $F$  and  $G$ , respectively. If

$$\frac{AM}{AB} = t,$$

find

$$\frac{EG}{EF}$$

in terms of  $t$ .

2. Let  $n \geq 3$  and consider a set  $E$  of  $2n - 1$  distinct points on a circle. Suppose that exactly  $k$  of these points are to be colored black. Such a coloring is "good" if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly  $n$  points from  $E$ . Find the smallest value of  $k$  so that every such coloring of  $k$  points of  $E$  is good.
3. Determine all integers  $n > 1$  such that

$$\frac{2^n + 1}{n^2}$$

is an integer.

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4. Let  $\mathbb{Q}^+$  be the set of positive rational numbers. Construct a function  $f : \mathbb{Q}^+ \rightarrow \mathbb{Q}^+$  such that

$$f(xf(y)) = \frac{f(x)}{y}$$

for all  $x, y$  in  $\mathbb{Q}^+$ .

5. Given an initial integer  $n_0 > 1$ , two players,  $\mathcal{A}$  and  $\mathcal{B}$ , choose integers  $n_1, n_2, n_3, \dots$  alternately according to the following rules:

Knowing  $n_{2k}$ ,  $\mathcal{A}$  chooses any integer  $n_{2k+1}$  such that

$$n_{2k} \leq n_{2k+1} \leq n_{2k}^2.$$

Knowing  $n_{2k+1}$ ,  $\mathcal{B}$  chooses any integer  $n_{2k+2}$  such that

$$\frac{n_{2k+1}}{n_{2k+2}}$$

is a prime raised to a positive integer power.

Player  $\mathcal{A}$  wins the game by choosing the number 1990; player  $\mathcal{B}$  wins by choosing the number 1. For which  $n_0$  does:

- $\mathcal{A}$  have a winning strategy?
  - $\mathcal{B}$  have a winning strategy?
  - Neither player have a winning strategy?
6. Prove that there exists a convex 1990-gon with the following two properties:
- All angles are equal.
  - The lengths of the 1990 sides are the numbers  $1^2, 2^2, 3^2, \dots, 1990^2$  in some order.