

29th International Mathematical Olympiad

Canberra, Australia

Day I

1. Consider two coplanar circles of radii R and r ($R > r$) with the same center. Let P be a fixed point on the smaller circle and B a variable point on the larger circle. The line BP meets the larger circle again at C . The perpendicular l to BP at P meets the smaller circle again at A . (If l is tangent to the circle at P then $A = P$.)
 - (i) Find the set of values of $BC^2 + CA^2 + AB^2$.
 - (ii) Find the locus of the midpoint of BC .
2. Let n be a positive integer and let $A_1, A_2, \dots, A_{2n+1}$ be subsets of a set B . Suppose that
 - (a) Each A_i has exactly $2n$ elements,
 - (b) Each $A_i \cap A_j$ ($1 \leq i < j \leq 2n + 1$) contains exactly one element, and
 - (c) Every element of B belongs to at least two of the A_i .

For which values of n can one assign to every element of B one of the numbers 0 and 1 in such a way that A_i has 0 assigned to exactly n of its elements?

3. A function f is defined on the positive integers by

$$\begin{aligned}f(1) &= 1, & f(3) &= 3, \\f(2n) &= f(n), \\f(4n + 1) &= 2f(2n + 1) - f(n), \\f(4n + 3) &= 3f(2n + 1) - 2f(n),\end{aligned}$$

for all positive integers n .

Determine the number of positive integers n , less than or equal to 1988, for which $f(n) = n$.

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Day II

4. Show that set of real numbers x which satisfy the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths is 1988.

5. ABC is a triangle right-angled at A , and D is the foot of the altitude from A . The straight line joining the incenters of the triangles ABD , ACD intersects the sides AB , AC at the points K , L respectively. S and T denote the areas of the triangles ABC and AKL respectively. Show that $S \geq 2T$.
6. Let a and b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that

$$\frac{a^2 + b^2}{ab + 1}$$

is the square of an integer.