The 3rd Balkan Mathematical Olympiad

Bucuresti, Romania, 1986

1. A line passing through the incenter I of ABC, intersects the incircle at D and E, and the circumcircle at F and G. D is between I and F. Prove that DF·EG ≥ r^2, where r is he inradius. When does equality take place?

2. Let ABCD be a tetrahedron with points E, F, G, H, K, L on edges AB, BC, CA, DA, DB, DC. If AE·BE = BF·CF = CG·AG = DH·AH = DK·BK = DL·CL then E, F, G, H, K, L are on a sphere.

3. Let $a_1 = a$, $a_2 = b$, $a_{n+1} = \dfrac{a_n^2 + c}{a_{n-1}}$, where a, b, c are real numbers with ab different from 0 and $c > 0$. Prove that a_k is integer for all k if and only if $a, b, \dfrac{a^2 + b^2 + c}{ab}$ are all integers.

4. Triangle ABC has the property that there is a point P in its plane so that PAB, PBC, PCA have the same areas and perimeters. Prove that if P is inside ABC then ABC is equilateral, and if P is not inside ABC then ABC is right-angled.