16th BALKAN MATHEMATICAL OLYMPIAD
Ohrid, Macedonia, May 7th, 1999

1. Given an acute triangle ABC, let D be the midpoint of the arc BC of the circumcircle around the triangle ABC, not containing the point A. The points which are symmetric to D with respect to the line BC and the circumcenter O are denoted by E and F, respectively. Finally, let K be the midpoint of the segment EA. Prove that:

- The circle passing through the midpoints of the sides of the triangle ABC, also passes through K;
- The line passing through K and the midpoint of the segment BC is perpendicular to the line AF.

2. Let $p > 2$ be a prime number such that $p = 2 \pmod 3$. Let $S = \{y^2 - x^3 - 1 \,/\, x, y$ are integers, $0 \le x, y \le p - 1\}$. Prove that at most p-1 elements of the set S are divisible by p.

3. Let ABC be an acute triangle, and let M, N, and P be the feet of the perpendiculars drawn from the centroid G of the triangle ABC towards its sides AB, BC, and CA, respectively. Prove that $\frac{4}{27} < \frac{area(MNP)}{area(ABC)} \le \frac{1}{4}$.

4. Let $x_0 = 0 \le x_1 \le x_2 \le x_3 \le ... \le x_n \le ...$ be a sequence of non-negative integers, such that for every k, $k \ge 0$, the number of terms of the sequence which are less than or equal to k is finite. Let this number be y_k. Prove that for all positive integers m, n we have:

$$\sum_{i=1}^{m} x_i + \sum_{i=1}^{n} y_i \ge (m+1)(n+1).$$