The 22nd Austrian–Polish Mathematics Competition
Austria, June 30 – July 2, 1999

1. Let n be a positive integer and $M = 1, 2, \ldots, n$. Find the number of ordered 6-tuples $(A_1, A_2, A_3, A_4, A_5, A_6)$ which satisfy the following two conditions:
a) sets $A_1, A_2, A_3, A_4, A_5, A_6$ (not necessarily different) are subsets of M
b) each element of M belongs either to exactly three subsets or to exactly six subsets or does not belong to any subset $A_1, A_2, A_3, A_4, A_5, A_6$.

2. Find the largest real number C_1 and the smallest real number C_2 such that for all real numbers a, b, c, d, e the following inequalities hold

$$C_1 < \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+e} + \frac{e}{e+a} < C_2 \,.$$

3. Let $n \geq 2$ be a given integer. Determine all systems of n functions $(f_1, \ldots, f_n)$ where $f_i : R \to R$ $i = 1, \ldots, n$ such that for all $x, y \in R$ the following equalities hold

$$\begin{aligned} f_1(x) - f_2(x) f_2(y) + f_1(y) &= 0 \\ f_2(x^2) - f_3(x) f_3(y) + f_2(y^2) &= 0 \\ \ldots\ldots & \\ f_k(x^k) - f_{k+1}(x) f_{k+1}(y) + f_k(y^k) &= 0 \\ \ldots\ldots & \\ f_n(x^n) - f_1(x) f_1(y) + f_n(y^n) &= 0 \,. \end{aligned}$$

4. Through a point P, which lies inside the triangle ABC, are drawn three straight lines k, l, m in such a way that:
a) k meets the lines AB and AC in A_1 and in A_2 ($A_1 \neq A_2$) respectively and $PA_1 = PA_2$,
b) similarly l meets the lines BC and BA in B_1 and in B_2 ($A_1 \neq A_2$) respectively and $PB_1 = PB_2$,
c) and similarly m meets the lines CA and CB in C_1 and in C_2 ($C_1 \neq C_2$) respectively and $PC_1 = PC_2$.
Prove that the lines k, l, m are uniquely determined by the conditions a),b),c). Find the point P (and prove that there exists exactly one such point) for which the triangles AA_1A_2, BB_1B_2, and CC_1C_2 have the same area.

5. A sequence of integers (a_n) satisfies the following recursive equation

$$a_{n+1} = a_n^3 + 1999 \quad \text{for } \ n = 1, 2, \ldots \,.$$

Prove that there exists at most one such n for which a_n is the square of an integer.

6. Solve the following system of equations

$$x_n^2 + x_n x_{n-1} + x_{n-1}^4 = 1 \quad \text{for} \quad n = 1, 2, \ldots, 1999$$
$$x_0 = x_{1999}$$

in the set of nonnegative real numbers.

7. Find all pairs (x, y) of positive integers such that

$$x^{x+y} = y^{y-x} \,.$$

8. Let g be a given straight line and let the points P, Q, R all lie on the same side of the line g. The points M, N lie on the line g and satisfy $PM \perp g$ and $QN \perp g$. The point S lies between the lines PM and QN and additionaly satisfies $PM = PS$ and $QN = QS$. The bisectors of SM and SN meet in the point R. The line RS intersects the circumcircle of the triangle PQR in $T \neq R$. Prove that S is the midpoint of the segment RT.

9. A point in the plane with both integer cartesian coordinates is called a lattice point. Consider the following one player game. A finite set of selected lattice points and finite set of selected segments is called a position in this game if the following hold:

a) the endpoints of each selected segment are lattice points,

b) each selected segment is parallel to a coordinate axis, or to the line $y = x$, or to the line $y = -x$,

c) each selected segment contains exactly five lattice points and all of them are selected,

d) each two selected segments have at most one common point.

A move in this game consists of selecting a lattice point and a segment such that the new set of selected lattice points and selected segment is a position. Prove or disprove that there exists an initial position such that the game has infinitely many moves.