The 18-th Austrian–Polish Mathematics Competition
Austria, June 28–30, 1995

1. Let n be a natural number. Determine all solutions $(a_1, \ldots, a_n)$ of the following system of equations

$$\begin{cases} a_3 = a_2 + a_1 \\ a_4 = a_3 + a_2 \\ \quad \cdots \\ a_n = a_{n-1} + a_{n-2} \\ a_1 = a_n + a_{n-1} \\ a_2 = a_1 + a_n, \end{cases}$$

where $a_1, \ldots, a_n$ are real numbers.

2. Let A_1, A_2, A_3, A_4 be four distinct points on the plane and let $X = \{A_1, A_2, A_3, A_4\}$. Prove that there exists a subset Y of X with the following property:

There does not exist a filled circle K, such that $K \cap X = Y$.

Note: All the points on the circle belong to the filled circle.

3. Let $P(x) = x^4 + x^3 + x^2 + x + 1$. Prove that there exist two polynomials $Q(y)$ and $R(y)$ with degree greater than or equal to 1, with integer coefficients, such that for all y

$$Q(y) \cdot R(y) = P(5y^2).$$

4. Determine all polynomials $P(x)$ with real coefficients, such that for all $x \neq 0$

$$\big(P(x)\big)^2 + \big(P(1/x)\big)^2 = P(x^2)\,P(1/x^2).$$

5. Given is an equilateral triangle ABC. Let A_1 be the midpoint of BC, B_1 the midpoint of AC and C_1 the midpoint of AB. Let p, q, r be three distinct parallel lines, such that the point A_1 lies on the line p, the point B_1 lies on the line q and the point C_1 lies on the line r. The line p intersects the line B_1C_1 in point A_2, the line q intersects the line A_1C_1 in point B_2 and the line r intersects the line A_1B_1 in point C_2. Prove that the lines AA_2, BB_2, CC_2 intersect in one point D and the point D lies on the circumcircle of the triangle ABC.

6. The Alpine Club, consisting of n members, organizes four mountain trips for their members. Let E_1, E_2, E_3, E_4 be the teams participating in these trips. In how many ways one can choose the teams with the condition that

$$E_1 \cap E_2 \neq \emptyset, \qquad E_2 \cap E_3 \neq \emptyset, \qquad E_3 \cap E_4 \neq \emptyset\,?$$

7. For every integer c we consider the equation

$$3y^4 + 4cy^3 + 2xy + 48 = 0.$$

In this equation the unknowns x and y are integers. Determine all integers c, such that the number of the integral solutions (x, y), satisfying the additional conditions (A) and (B), is maximal.

(A) The number $|x|$ is a square of an integer.

(B) The number y is squarefree, i.e. there doesn't exist a prime p, such that p^2 is a divisor of y.

8. Consider the cube whose eight vertices have the coefficients $(\pm1, \pm1, \pm1)$, i.e. the set of the points

$$\{(x, y, z)\colon\ |x| \le 1,\ |y| \le 1,\ |z| \le 1\}.$$

Let $V_1, \ldots, V_{95}$ be any points of this cube. Denote by v_i the vector from the point $(0, 0, 0)$ to the point V_i. Consider the 2^{95} vectors of the form $s_1v_1 + s_2v_2 + \cdots + s_{95}v_{95}$, where $s_i = +1$ or $s_i = -1$.

a) Let $d = 48$. Prove that among these vectors there exists a vector $w = (a, b, c)$, such that $a^2 + b^2 + c^2 \le d$.

b) Find a number $d < 48$ with the same property.

Note: The less the number d is obtained, the higher the solution will be scored.

9. Prove that for all natural numbers $n, m \ge 1$ and for all positive real numbers x, y the following inequality holds:

$$(n-1)(m-1)(x^{n+m} + y^{n+m}) + (n+m-1)(x^n y^m + x^m y^n) \ge nm(x^{n+m-1}y + xy^{n+m-1}).$$