1^{st} Romanian Selection Test for the 6^{th} JBMO

1. For a positive integer n let $f(n) = \frac{4n+\sqrt{4n^2-1}}{\sqrt{2n+1}+\sqrt{2n-1}}$. Calculate: $f(1) + f(2) + ... + f(40)$.

2. Let n, p, k be nonnegative integers so that p is prime, $k < 1000$ and $\sqrt{k} = n\sqrt{p}$.

   (a) Prove that if the equation $\sqrt{k + 100x} = (n+x)\sqrt{p}$ has an integer solution different from 0 then $p|10$.

   (b) In that case find the number of solutions of the equation (that is, when $p = 2$ or $p = 5$).

3. Consider a $1 \times n$ rectangle made out of n tiles. A "pavement" is a coloring of each of the n tiles with one of 4 possible colors so that no two consecutive tiles have the same color.

   (a) What is the number of the distinct symmetrical "pavements" (a symmetrical "pavement" is a pavement for which tiles symmetrical with respect to the center have the same color).

   (b) What is the number of distinct "pavements" so that in any block of three consecutive tiles no two tiles have the same color?

4. Let $ABCD$ be a parallelogram with the center O. Let M and N be the midpoints of OB and CD, respectively. Prove that if triangles and are similar then $ABCD$ is a square.

2^{nd} Romanian Selection Test for the 6^{th} JBMO

1. A unit square is divided naturally into 9 congruent squares (of side $\frac{1}{3}$). The central square is colored. We call this procedure P. For each of the remaining 8 squares apply procedure P. For each of the 64 obtained squares apply procedure P and so on. Prove that after 1000 applications of procedure P the area colored exceeds 0,999.

2. Find all positive integers a, b, c, d so that $a + b + c + d - 3 = ab = cd$.

3. Let ΔABC be an isosceles triangle ($AB = AC$) so that $m(\angle BAC) = 20°$. Let M be the projection of point C to AB and N a point on side AC so that $CN = \frac{BC}{2}$. Find $m(\angle AMN)$.

4. Let $ABCD$ be a unit square and M, N points in its interior. We know that no vertex of the square lies on line MN. Let $s(M, N)$ be the smallest area of a triangle with vertices in the set A, B, C, D, M, N. Find the smallest real number k so that for any points M, N with the mentioned property we have $s(M, N) \leq k$.

3^{rd} Romanian Selection Test for the 6^{th} JBMO

1. Let n be an even nonnegative integer and a, b nonnegative co prime integers. Find a and b is $a + b | a^n + b^n$.

2. Let $ABCD$ be a convex quadrilateral and O the point of intersection of its diagonals. The measure of the angle between the two diagonals is m. For any angle xOy of measure m, the area inside the angle that is in the interior of the quadrilateral is constant. Prove that $ABCD$ is a square.

3. An equilateral triangle of side 10 is divided into 100 unit equilateral triangles by lines parallel to the sides of the triangle. Find the number of (not necessarily unit) equilateral triangles in the configuration described above so that the sides of the triangle are parallel to the sides of the initial one.

4. If $a, b, c \in (0, 1)$ then $\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1$.

Thanks to **Gabriel Bucur.**

4^{th} Romanian Selection Test for the 6^{th} JBMO

1. Let a be an integer number. Prove that for every real number x so that $x^2 < 3$ the numbers $\sqrt{3 - x^2}$ and $\sqrt[3]{a - x^3}$ are not simultaneously rational.

2. The last 4 digits of a perfect square are equal. Prove that they are all 0.

3. Two circles $\mathbf{C}_1(O_1)$ and $\mathbf{C}_2(O_2)$ are given so that $O_2 \in \mathbf{C}_1$ and M is a point on $\mathbf{C}_1$, MO_1O_2. The tangents from M to $\mathbf{C}_2$ intersect $\mathbf{C}_1$ in the points A and B. Prove that the tangents from A and B to $\mathbf{C}_2$ which don't pass trough M, are concurrent in a point of $\mathbf{C}_1$.

4. Five points from a plane are given so that every 3 of them form a triangle with the area at least 2. Prove that there are 3 points which form a triangle with the area at least 3.

5^{th} Romanian Selection Test for the 6^{th} JBMO

1. Solve the following equation in the set of positive integers: $x^n + y^n = 2^{2002}$, $n > 1$.

2. Consider $n > 2$ concentric circles and two lines d_1 and d_2 concurrent in a point P interior to these circles. The semi lines determined by the point P on d_1 intersect the circles in the points $A_1, A_2, ..., A_n$ and $A'_1, A'_2, ..., A'_n$ respectively and the semi lines determined by the point P on d_2 intersect the circles in the points $B_1, B_2, ..., B_n$ and $B'_1, B'_2, ..., B'_n$(the points with the same indices lie on the same circle). Prove that if

the little arcs A_1B_1 and A_2B_2 are congruent than all the little arcs A_iB_i and $A'_iB'_i$ are congruent for all $i = 1, 2, ..., n$.

3. Let ΔABC be a triangle in which we denote $a = BC, b = AC, c = AB$. In the same part of BC as A we consider the points D and E so that $BD = c, CE = b$ and the area of the quadrilateral $BCED$ is maximum. Let F be the midpoint of DE and $FB = x$. Prove that $FC = FB$ and $4x^3 = (a^2 + b^2 + c^2)x^2 + abc$.

4. Let p and q be two distinct primes. Prove the existence of the positive integers a and b, so that the arithmetic mean of all the natural divisors of the number $n = p^a q^b$ is an integer number.

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