1st Romanian Selection Test for the 43rd IMO
Ramnicu Valcea - March 21st, 2002

1. Find all sets *A*, *B* which fulfill the following conditions:
a) *A*È *B* = **Z** ;
b) if *x* Î *A* then *x*-1 Î *B*;
c) if *x*, *y* Î *B* then *x* + *y* Î *A*.
2. Given the sequence $(a_n)_{n ³ 0}$ defined by $a_0 = a_1 = 1$ and $a_{n+1} = 14a_n - a_{n-1}$ for all *n* ³ 1, prove that for any *n* Î **N** $2a_n$ -1 is a perfect square.
3. Let ΔABC be an acute-angled triangle. Consider the midpoints *M*, *N* of [*AB*] and [*AC*] and let *P* be the point where the perpendicular from *N* on *BC* intersects *BC*. Now, let A_1 be the midpoint of *MP*. Construct points B_1, C_1 similarly. Prove that if the lines AA_1, BB_1, CC_1 intersect in one point, then ΔABC is isosceles.
4. For any *n* positive integer let *f*(*n*) be the number of choices of signs + and - in the expression: $E = \pm 1 \pm 2 \pm \cdots \pm n$, so that the value of the expression is null. Prove that:
i) $f(n) = 0$ for $n \equiv 1, 2 \pmod 4$;
ii) for $n \equiv 0, 3 \pmod 4$ we have:

$$\frac{1}{2}(\sqrt{2})^n \le f(n) \le 2^n - 2^{\left[\frac{n}{2}\right]+1}.$$

2nd Romanian Selection Test for the 43rd IMO
Bucharest –April 13th, 2002

1. Let *ABCD* be a unit-length sided square. For any two distinct points *M*, *N* found in the interior of the square such that the line *MN* doesn't pass through any of the square's vertices, we denote by *s*(*M*, *N*) the smallest area of a triangle with vertices in the set *{ A, B, C, D, M, N }* . Determine the smallest real *k* for which *s*(*M*, *N*) £ *k*, for all *M*, *N* with the above properties.
2. Let *P*, *Q* Î **Z** [*X*] of degree *p*, *q* respectively, with coefficients in the set {1, 2002}. Prove that if *P* | *Q* then *p* + 1 | *q* + 1.
3. Let *a*, *b* be two positive reals. For every positive integer *n*, denote by x_n the digit-sum of [*an* + *b*] (in decimal representation). Prove that the sequence $(x_n)_{n ³ 1}$ contains a constant sub-sequence.
4. Four official languages are used at an international conference, and any two participants can communicate one with each other in one of these official languages. Prove that there is a language that is spoken by at least 60% of the participants.

3rd Romanian Selection Test for the 43rd IMO
Bucharest –April 14th, 2002

1. Let *ABCDE* be a pentagon inscribed in a circle of center *O*, having $m(\angle B) = 120^o$, $m(\angle C) = 120^o$, $m(\angle D) = 130^o$ and $m(\angle E) = 100^o$. Prove that the intersection point of the lines *BD* and *CE* lies on the diameter *AO*.

2. Let $n \geq 4$ be an integer and $a_1, a_2, \ldots, a_n$ positive reals such that $a_1^2 + a_2^2 + \ldots + a_n^2 = 1$.
Prove that:

$$\frac{a_1}{a_2^2+1} + \frac{a_2}{a_3^2+1} + \ldots + \frac{a_n}{a_1^2+1} \geq \frac{4}{5}\left(a_1\sqrt{a_1} + a_2\sqrt{a_2} + \ldots + a_n\sqrt{a_n}\right)^2$$

3. Let n be an even positive integer and let S be the set of positive integers a which fulfill: $1 < a < n$ and $a^{a-1} - 1$ does not divide n. Prove that if $S = \{ n - 1 \}$ then there is a prime p such that $n = 2p$.
4. Consider a function $f: Z \to \{1, 2, \ldots, n\}$ with the property that: : $f(x) \neq f(y)$, $\forall x, y$ Î Z such that $| x - y |$ Î $\{ 2, 3, 5 \}$. Prove that n ³ 4.

4th Romanian Selection Test for the 43rd IMO
Bucharest –June 1st, 2002

1. Let $(a_n)_{n ³ 1}$ be sequence of positive integers such that a_{n+1} is the smallest prime divisor of $a_n + a_{n-1}$ $\forall$ n ³ 2. Prove that a real number x, whose decimal digits are the digits of a_1 , a_2 ,…, a_n ,…, placed in this order, is a rational number.
2. Determine the smallest positive number r for which the following property is true: no matter how you construct four disjoint circles with the centers in the vertices of a unit-length square and the sum of their radii is r, there exists an equilateral triangle with vertices in the interior of three of the circles.
3. After elections, each senator in the CaliMass Congress has an absolute rating (the number of voters for example - anyway a positive number). Each of these senators enters in a committee in which he obtains a relative rating. The relative rating of a senator is the ratio between his absolute rating and the sum of all ratings in his committee (including his rating). A senator has the right to move from his committee to another committee if in this way his relative rating grows. In each day only one senator is allowed to change his committee. Prove that the senators cannot move forever.

5th Romanian Selection Test for the 43rd IMO
Bucharest –June 2nd, 2002

1. Let m and n be two positive integers having different parities and fulfilling $m < n < 5m$. Prove that the numbers $\{1, 2, 3, \ldots, 4mn \}$ can be paired such that in every pair the sum of the two numbers composing that pair is a perfect square.
2. Given are a circle K and a triangle ΔABC inscribed in it such that AC ¹ BC, $AB < AC$. The tangent from A at K intersects the line BC in D. Consider the circle K_1, which is tangent to K and the line segments $[AD]$ and $[BD]$. Denote by M the point of tangency of K_1 with $[BD]$. Prove that $AC = MC$ if and only if $\angle MAB = \angle MAD$.
3. Given are a set of np cards, n ³ 2, colored with colors $c_1, c_2, \ldots, c_n$ and numbered with the numbers $1, 2, \ldots, p$. There are n players and each player receives p cards from the set. They will play a game with p rounds, using the following rules:
A: first player begins with a card and everybody else has to play a card of the same color; in case they have no card of that color they can play any other card.

B: the player wins the first round with the biggest card number, having the same color as the card played by the first player.
C: the player who won the last round begins a new one with a card chosen by him from his remaining set of cards and the game continues as described in A and B.
D: all played card are being put aside after each round.
Prove that if all the n cards having the number 1 have won rounds then $p \geq 2n$.