ROMANIAN MATHEMATICAL OLYMPIAD
FINAL ROUND - 2002

IX$^{\text{th}}$ FORM

Here are my solutions:

1. Let $a, b, c \in \mathbf{R}_+$ so that $ab + bc + ca = 1$. Prove that:

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \sqrt{3} + \frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}.$$

Proof: Write the inequality in the form:

$$\frac{1-ab}{a+b} + \frac{1-bc}{b+c} + \frac{1-ca}{c+a} \geq \sqrt{3}$$

From the fact that $ab + bc + ca = 1$ we can write that:

$$\frac{c(a+b)}{a+b} + \frac{a(b+c)}{b+c} + \frac{b(c+a)}{c+a} \geq \sqrt{3} \Leftrightarrow a+b+c \geq \sqrt{3} \Leftrightarrow (a+b+c)^2 \geq 3$$

Again using $ab + bc + ca = 1$ we have to prove that

$$(a+b+c)^2 \geq 3(ab+bc+ca) \Leftrightarrow a^2+b^2+c^2 \geq ab+bc+ca,$$

which is known.

2. Let ΔABC with $m(\angle BAC) = 90°$ and $M \in (AB)$ so that $\frac{AM}{MB} = 3\sqrt{3} - 4$. Let N be the midpoint of GI and M' a point on the line MN so that $MN = NM'$ and $N \in [MM']$. Determine $m(\angle ABC)$ knowing that $M' \in AC$.

Proof: For every point P in the plane the following are true:

$$3\overrightarrow{PG} = \overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}; \overrightarrow{PI} = \frac{a\overrightarrow{PA} + b\overrightarrow{PB} + c\overrightarrow{PC}}{a+b+c}$$

Now letting $P = A$ we have that:

$$\overrightarrow{AG} = \frac{\overrightarrow{AB} + \overrightarrow{AC}}{3}; \overrightarrow{AI} = \frac{b\overrightarrow{AB} + c\overrightarrow{AC}}{a+b+c}.$$

Because N is the midpoint of GI and MM' we have that:

$$\overrightarrow{AN} = \frac{\overrightarrow{AG} + \overrightarrow{AI}}{2} = \frac{\overrightarrow{AM} + \overrightarrow{AM'}}{2}.$$

From the fact that $M' \in AC$ we have obviously $\overrightarrow{AM'} = l\overrightarrow{AC}$, l is a constant. Denoting $\frac{AM}{AB} = k$ we have that $\overrightarrow{AM} = k\overrightarrow{AB}$. Now it's obvious that:

$$\overrightarrow{AG} + \overrightarrow{AI} = \overrightarrow{AM} + \overrightarrow{AM'} \Leftrightarrow \left(\frac{1}{3} + \frac{b}{a+b+c}\right)\overrightarrow{AB} + \left(\frac{1}{3} + \frac{c}{a+b+c}\right)\overrightarrow{AC} = k\overrightarrow{AB} + l\overrightarrow{AC}$$

Because $\overrightarrow{AB}$ and $\overrightarrow{AC}$ aren't collinear we have that:

$$\frac{1}{3} + \frac{b}{a+b+c} = k$$

$$\frac{1}{3} + \frac{c}{a+b+c} = l.$$

We are interested only in the first relation. We can write it in the following ways:

$$(1-3k)a = (4-3k)b + (1-3k)c \Leftrightarrow (1-3k)^2a^2 = [(4-3k)b + (1-3k)c]^2 \Leftrightarrow$$

$$(1-3k)^2(b^2+c^2) = [(4-3k)b + (1-3k)c]^2.$$

Using some computations (for example, divide the relation by b) and the fact that $k = \frac{3\sqrt{3}-4}{3\sqrt{3}-3}$ we can find out that

$$\tan(\angle B) = \frac{b}{c} = \frac{1}{\sqrt{3}} \Leftrightarrow m(\angle B) = 30^\circ$$

3. Let $n, k \in \mathbf{N^*}$ so that $n > 2$. Prove that the equation:

$$x^n - y^n = 2^k$$

has no solutions in the set of positive integers.

Proof: I haven't made this problem so I'll give you the official solution. We will prove by contradiction. Let $n_0 > 2$ the minimum for which exists $m \in \mathbf{N}^*$ so that $x^{n_0} - y^{n_0} = 2^m$. If n_0 would be even, $n = 2a$ we would have that $(x^a - y^a)(x^a + y^a) = 2^m$ so $x^a - y^a = 2^l$ thus n_0 isn't minimum, contradiction. So n is odd. Let $A = \{p \in \mathbf{N}^* |$ exists $x, y \in \mathbf{N}^*, x^{n_0} - y^{n_0} = 2^p\}$. Let now p_0 be the smallest element of A. If $x^{n_0} - y^{n_0} = 2^{p_0}$ then it's obvious that x, y have the same parity. From the formula $(x-y)(x^{n_0-1} + ... + y^{n_0-1} = 2^{p_0})$ we have that x, y are even. But now $x = 2x_1, y = 2y_1$ implies that $x^{n_0} - y^{n_0} = 2^{p_0 - n_0}$ which contradicts the minimality of p_0 or $x_1^{n_0} - y_1^{n_0} = 1$ which hasn't any solutions in $\mathbf{N}^*$ for $n_0 > 2$.

4. Determine the functions $f : \mathbf{N} \to \mathbf{N}$ so that

$$f(3x + 2y) = f(x)f(y),$$

$\forall x, y \in \mathbf{N}$.

Proof: Letting $x = y = 0$ we have that $f(0) = f^2(0)$ so $f(0) = 0$ or $f(0) = 1$. $f(0) = 0$ implies that $f(3x) = f(2y) = 0$. Now it's easy to see that $f(2x) = f(3x) = f(7x) = 0$, $\forall x \in \mathbf{N}$ and letting $x = y$ we have that $f(5x) = f^2(x) \Rightarrow f(25x) = f^4(x)$. But $f(25x) = f(3 \cdot 7x + 2 \cdot 2x) = f(7x)f(2x) = 0$ so $f(x) = 0$, $\forall x \in \mathbf{N}$. If $f(0) = 1$ it's obvious that $f(2y) = f(y), f(3x) = f(x)$ (by letting $x = 0, y = 0$ respectively). Together with $f(1) = a$ this implies that $f(2) = a, f(3) = a, f(5) = f^2(1) = a^2, f(7) = f(2)f(1) = a^2, f(25) = f^2(5) = a^4 = f(3 \cdot 7 + 2 \cdot 2) = a^3$ so $a^3 = a^4 \Rightarrow a = 0$ or $a = 1$. So we obtain the functions $f(x) = 1, \forall x \in \mathbf{N}$ or $f(x) = \begin{cases} 1, \text{ if } x = 0 \\ 0, \text{ if } x > 0 \end{cases}$.