ROMANIAN MATHEMATICAL OLYMPIAD
FINAL ROUND - 2002

IX$^{\text{th}}$ FORM

1. Let $a, b, c \in \mathbf{R}_+$ so that $ab + bc + ca = 1$. Prove that:
$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \sqrt{3} + \frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}.$$

2. Let ΔABC with $m(\angle BAC) = 90°$ and $M \in (AB)$ so that $\frac{AM}{MB} = 3\sqrt{3} - 4$. Let N be the midpoint of GI and M' a point on the line MN so that $MN = NM'$ and $N \in [MM']$. Determine $m(\angle ABC)$ knowing that $M' \in AC$.

3. Let $n, k \in \mathbf{N^*}$ so that $n > 2$. Prove that the equation:
$$x^n - y^n = 2^k$$
has no solutions in the set of positive integers.

4. Determine the functions $f : \mathbf{N} \to \mathbf{N}$ so that
$$f(3x + 2y) = f(x)f(y),$$
$\forall x, y \in \mathbf{N}$.

X$^{\text{th}}$ FORM

1. Let X, Y, Z, T distinct points in space. We say that the segments $[XY]$ and $[ZT]$ are *connected* if there exists a point O in the plane so that ΔOXY and ΔOZT are isosceles and right-angled, with $m(\angle O) = 90°$. Let $ABCDEF$ be a regular hexagon in which both $[AB]$, $[CD]$ and $[BD]$, $[EF]$ are *connected*. Prove that the points A, C, D, F are the vertices of a parallelogram and $[BC]$, $[AE]$ are also *connected*.

2. Determine the polynomials $f, g \in \mathbf{R}[X]$, knowing that:
$$(x^2 + x + 1)f(x^2 - x + 1) = (x^2 - x + 1)g(x^2 + x + 1),$$
$\forall x \in \mathbf{R}$.

3. Let $a, b, c, d, e \in [-2, 2]$, for which:
$$\begin{cases} a + b + c + d = 0 \\ a^3 + b^3 + c^3 + d^3 + e^3 = 0 \\ a^5 + b^5 + c^5 + d^5 + e^5 = 10 \end{cases}.$$

4. Let $I \subseteq \mathbf{R}$ an interval and $f : I \to \mathbf{R}$ a function so that:

$$| f(x) - f(y) | \leq | x - y |, \forall x, y \in I.$$

Prove that f is monotone on I iff $\forall x, y \in I$, we have: $f(x) \leq f(\frac{x+y}{2}) \leq f(y)$ or $f(y) \leq f(\frac{x+y}{2}) \leq f(x)$.

XI$^{\text{th}}$ FORM

1. In the cartesian plane xOy consider the hyperbola

$$\Gamma = \{M(x, y) \in \mathbf{R}^2 \mid \frac{x^2}{4} - y^2 = 1\}$$

and a conic Γ' different from Γ. Denote by $n(\Gamma, \Gamma')$ the maximum number of pairs of points $(A, A') \in \Gamma \times \Gamma'$ for which $AA' \leq BB'$, $\forall (B, B') \in \Gamma \times \Gamma'$. For every $p \in \{0, 1, 2, 4\}$ write the cartesian equation of the conic Γ' for which $n(\Gamma, \Gamma') = p$. Justify your answer.

2. Let $f : \mathbf{R} \to \mathbf{R}$ which has a limit in every real point and hasn't got a local extreme point. Prove that

   (a) f is continuous;

   (b) f is strictly monotone.

3. Let $A \in \mathcal{M}_4(\mathbf{C})$ different from the 0 matrix.

   (a) If $rang(A) = r < 4$, prove that there exists $U, V \in \mathcal{M}_4(\mathbf{C})$, invertible, so that:

$$UAV = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

   (b) Prove that if $rang(A) = rang(A^2) = k$ then $rang(A^n) = k$, $\forall n \geq 3$.

4. Let $f : [0, 1] \to [0, 1]$ be a continuous and bijective function. Determine the set:

$$A = \{f(x) - f(y) | x, y \in [0, 1] \setminus \mathbf{Q}\}.$$

(Consider known that: *an injective function* $f : \boldsymbol{R} \setminus \boldsymbol{Q} \to \boldsymbol{Q}$ *doesn't exist.)*

XII$^{\text{th}}$ FORM

1. Consider a ring A.

   (a) Prove that the set $Z(A) = \{a \in A | ax = xa, \forall x \in A\}$ is a sub-ring of A.

(b) Prove that if every commutative sub-ring of A is a field.

(A sub-ring is a subset $B \subset A$ which fulfils the following conditions: $x, y \in B$ implies $xy, x - y \in B$ and $1 \in B$ is the neutral element of multiplication in A.)

2. Let $f : [0, 1] \to \mathbf{R}$, integrable, so that:

$$0 <\mid \int_0^1 f(x)dx \mid\leq 1.$$

Prove that there exists $x_1 \neq x_2$, $x_1, x_2 \in [0, 1]$, so that:

$$\int_{x_1}^{x_2} f(x)dx = (x_1 - x_2)^{2002}.$$

3. Let $f : \mathbf{R} \to \mathbf{R}$ a continuous and limited function. If

$$x \int_x^{x+1} f(t)dt = \int_0^x f(t)dt, \forall x \in \mathbf{R},$$

prove that f is constant.

4. Let K be a field with $q = p^n$ elements, p prime and $n \geq 2$. For $a \in K$, arbitrary, define the polynomial $f_a = X^p - X + a$. Prove that:

(a) f_1 divides the polynomial $f = (X^p - X)^q - (X^p - X)$;

(b) f_a has at least p^{n-1} irreducible divisors in $K[X]$, un-associated in divisibility.

(Consider known that: *every finite field is commutative.*)

Compiled by **Călin Popa**.