ROMANIAN MATHEMATICAL OLYMPIAD
FINAL ROUND - 2001

IX$^{\text{th}}$ FORM

1. Let A be a set of real numbers so that:

   (a) $1 \in A$;

   (b) $x \in A \Rightarrow x^2 \in A$;

   (c) $x^2 - 4x + 4 \in A \Rightarrow x \in A$.

   Prove that $2000 + \sqrt{2001} \in A$.

2. Let ΔABC with $m(\angle BAC) = 90°$ and $D \in (AC)$ the leg of the bisector of the $\angle B$. Prove that $BC - BD = 2AB \Leftrightarrow \frac{1}{BD} - \frac{1}{BC} = \frac{1}{2AB}$.

3. Let $n \in \mathbf{N}^*$ and $v_1, v_2, ..., v_n$ vectors in the plane with lengths less or equal to 1. Prove that there exists the numbers $\epsilon_1, \epsilon_2, ..., \epsilon_n \in \{-1, 1\}$ so that:$|\epsilon_1 v_1 + \epsilon_2 v_2 + ... + \epsilon_n v_n| \leq \sqrt{2}$.

4. Determine $x, y, z \in \mathbf{Q}_+$ so that: $x + \frac{1}{y}$, $y + \frac{1}{z}$ and $z + \frac{1}{x}$ are positive integers.

X$^{\text{th}}$ FORM

1. Let $a, b \in \mathbf{C}^*$ and z_1, z_2 the roots of the polynomial $X^2 + aX + b = 0$. Prove that $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \exists \lambda \in \mathbf{R}$ so that $a^2 = \lambda b$.

2. In the tetrahedron $OABC$ let $\alpha = m(\angle BOC), \beta = m(\angle COA), \gamma = m(\angle AOB)$. Prove that: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma < 1 + 2\cos\alpha\cos\beta\cos\gamma$.

3. Let $k, m \in \mathbf{N}^*$, $k < m$ and let M be a set with m elements. Prove that the maximum number of subsets $A_1, A_2, ..., A_n$ of M for which $A_i \cap A_j$ has at most k elements, for every $1 \leq i \leq j \leq p$ is:

$$p_{max} = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + ... + \binom{m}{k+1}.$$

4. Let $n \geq 2$ be an even positive integer and a, b real numbers so that $b^n = 3a + 1$. Prove that the polynomial $P(X) = (X^2 + X + 1)^n - X^n - a$ is divisible by $Q(X) = X^3 + X^2 + X + b$ iff $b = 1$.

XI$^{\text{th}}$ FORM

1. Let $f : \mathbf{R} \to \mathbf{R}$ continuous on $\mathbf{R}$, derivable on $\mathbf{R}_0$, having its lateral derivatives finites in x_0. Prove that there exists the function $g : \mathbf{R} \to \mathbf{R}$, derivable, a function $h : \mathbf{R} \to \mathbf{R}$, of first degree, and $\alpha \in \{-1, 0, 1\}$ so that: $f(x) = g(x) + \alpha|h(x)|$.

2. Let $A \in \mathcal{M}_n(\mathbf{C})$ be a matrix with $rangA = r$, $n \geq 2, 1 \leq r \leq n-1$.
   (a) Prove that there exists $B \in \mathcal{M}_{n,r}(\mathbf{C}), C \in \mathcal{M}_{r,n}(\mathbf{C})$, with $rangB = rangC = r$, so that $A = BC$.
   (b) Prove that A verifies a polynomial equation of degree r+1, with complex coefficients.

3. Let $f : \mathbf{R} \to [0, \infty)$ be a function so that $|f(x) - f(y)| \leq |x - y|, \forall x, y \in \mathbf{R}$. Prove that:
   (a) if $\lim\limits_{n\to\infty} f(x+n) = \infty, \forall x \in \mathbf{R}$, then $\lim\limits_{x\to\infty} f(x) = \infty$;
   (b) if $\lim\limits_{n\to\infty} f(x+n) = a, a \in [0, \infty), \forall x \in \mathbf{R}$ then $\lim\limits_{x\to\infty} f(x) = a$.

4. The function $f : [0, 1] \to \mathbf{R}$ is continuous and has the propriety:
$$\lim_{n\to\infty} n\left(f\left(x + \frac{1}{n}\right) - f(x)\right) = 0, \forall x \in [0, 1).$$
   Prove that:
   (a) $\forall \epsilon > 0$ and $\lambda \in (0, 1)$ we have: $\sup\{x \in [0, \lambda) \,|\, |f(x) - f(0)| \leq \epsilon x\} = \lambda$;
   (b) f is constant.

XII$^{\text{th}}$ FORM

1. (a) Let $P(X) = X^5 \in \mathbf{R}[X]$. Prove that $\forall \alpha \in \mathbf{R}^*$ the polynomial $P(X + \alpha) - P(X)$ has no real roots.
   (b) Let $P \in \mathbf{R}[X]$ be a polynomial of degree $n \geq 2$ with real and distinct roots. Prove that $\exists \alpha \in \mathbf{Q}^*$ so that the polynomial $P(X + \alpha) - P(X)$ has real roots.

2. Let A be a finite ring. Prove that there exists two natural numbers m, p, $m > p \geq 1$ so that $a^m = a^p, \forall a \in A$.

3. Let $f : [-1, 1] \to \mathbf{R}$ be a continuous function. Prove that:
   (a) if
$$\int_0^1 f(\sin(x + \alpha))dx = 0, \forall \alpha \in \mathbf{R},$$
   then $f(x) = 0, \forall x \in [-1, 1]$;

(b) if

$$\int_0^1 f(\sin nx)dx = 0, \forall n \in \mathbf{Z},$$

then $f(x) = 0, \forall x \in [-1, 1]$.

4. Let $f : [0, \infty) \to \mathbf{R}$ be a periodic function, of period 1, integrable on $[0, 1]$. For a sequence $(x_n)_{n\geq 0}$, $x_0 = 0$, strictly increasing and unlimited with $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$, denote $r(n) = \max\{k|x_k \leq n\}$.

   (a) Prove that:

   $$\lim_{n\to\infty} \sum_{k=1}^{r(n)} (x_k - x_{k-1}) f(x_k) = \int_0^1 f(x)dx.$$

   (b) Using possibly the result from (a) prove that:

   $$\lim_{n\to\infty} \frac{1}{\ln n} \sum_{k=1}^{n} \frac{f(\ln k)}{k} = \int_0^1 f(x)dx$$

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