ROMANIAN MATHEMATICAL OLYMPIAD
FINAL ROUND - 2000

IX$^{\text{th}}$ FORM

1. Let $p, q \in \mathbf{N},\ 1 \leq q \leq p$ and $a = \left(\sqrt{p^2+q}+p\right)^2$. Prove that a is irrational and $\{a\} > 3/4$. ($\{\cdot\}$ is the fractional part function).
2. Let $a, b \in (0,1)$ and two fixed points A, B. For every point $M \notin AB$ consider $P \in (AM)$, $N \in (BM)$ such that $BN = bBM$ and $AP = aAM$. Find the locus of M such that $AN = BP$.
3. Determine the maximum value M_n of the following expression:
$$E = x_1^2 + x_2^2 + ... + x_n^2 - x_1x_2 - x_2x_3 - ... - x_{n-1}x_n - x_nx_1,$$
when $x_1, x_2, ..., x_n \in [0,1]$, then find all cases when $E = M_n$.
4. For every point M on the side AB of a triangle ABC there exist two points $N \in (BC),\ P \in (CA)$ such that the centroid of the triangle MNP is I - the incenter of the triangle ABC. Prove that ABC is equilateral.

X$^{\text{th}}$ FORM

1. Let $x_1 = 3,\ x_{n+1} = [\sqrt{2}x_n]$. Find n if x_n, x_{n+1}, x_{n+2} is an arithmetic progression.
2. Let $z_1, z_2 \in \mathbf{C}^*$ be such that $z_1 \cdot 2^{|z_2|} + z_2 \cdot 2^{|z_1|} = (z_1+z_2)\cdot 2^{|z_1+z_2|}$. Prove that $z_1^6 = z_2^6$.
3. Let E be the projection of the vertex D of a thetraedron $ABCD$ on (ABC). Prove that the following assertions are equivalent:
a) $C = E$ or $CE \| AB$;
b) $\forall\, M \in [CD],\ [ABM]^2 = \dfrac{CM^2}{CD^2}[ABD]^2 + \left(1 - \dfrac{CM^2}{CD^2}\right)[ABC]^2$. ([XYZ] is the area of the triangle XYZ).
4. Let f be a polynomial of third degree with rational coefficients and let x_1, x_2, x_3 be its roots. If exist $a, b \in \mathbf{Q}^*$ different such that $ax_1 + bx_2 \in \mathbf{Q}$, prove that $x_1, x_2, x_3 \in \mathbf{Q}$.

XI$^{\text{th}}$ FORM

1. Denote $M = \{A \in \mathcal{M}_2(\mathbf{C}) \mid \det(A - zI_2) = 0 \Rightarrow |z| < 1\}$. Prove that if $A, B \in M$ and $AB = BA$, then $AB \in M$.

2. Let $x_0 \in \mathbf{R} \setminus \mathbf{Q}$ and for $n \in \mathbf{N}$ we have: $x_{n+1} \in \left\{ \frac{x_n+1}{x_n}, \frac{x_n+2}{2x_n-1} \right\}$. Is $(x_n)_{n\in N}$ convergent?

3. We say that $f : \mathbf{R}^2 \to \mathbf{R}$ is *olympic* if for every $A_1, ..., A_n \in \mathbf{R}^2$, $n \geq 3$, any two different with $f(A_1) = ... = f(A_n)$, it results that the n-gon $A_1...A_n$ is convex. Prove that if $P \in \mathbf{C}[X]$ is non-constant then the following assertions are equivalent:
a) $f(x, y) = |P(x + iy)|$ is olympic;
b) all roots of P are equal.

4. Let $f : \mathbf{R} \to \mathbf{R}$ be a function with the intermediate value property such that $\lim\limits_{x\to\infty} f(f(x)) = \infty$ and $\lim\limits_{x\to-\infty} f(f(x)) = -\infty$. Prove that:
a) there exist the limits: $l_1 = \lim\limits_{x\to-\infty} f(x)$ and $l_2 = \lim\limits_{x\to\infty} f(x)$;
b) can be l_1 and l_2 both finite?

XII$^{\text{th}}$ FORM

1. Let $a > 1$ and $f : [1, \infty) \to \mathbf{R}$ be continuous such that $\lim\limits_{x\to\infty} xf(x) = l \in \mathbf{R}$.
a) Prove that the limits $l_1 = \lim\limits_{t\to\infty} \int_1^t \frac{f(x)}{x}\, dx$ and $l_2 = \lim\limits_{t\to\infty} t \int_1^a f(x^t)\, dx$ exist and $l_1 = l_2$.

2. Let $n \in \mathbf{N}$, $n \geq 2$. For every division $0 = x_0 < x_1 < ... < x_n = 1$ define

$$\mathcal{A}(x_0, ..., x_n) = \{f : [0, 1] \to \mathbf{R} \mid f\text{- derivable},$$

$$\int_0^1 |f(x)|\, dx = 1,\ f(x_k) = 0,\ k = \overline{0, n}\}.$$

Prove that there exists a division $a_0 < ... < a_n$ such that

$$\sup_{x\in[0,1]} |f'(x)| > 4n,\ \ \forall f \in \mathcal{A}(a_0, ..., a_n).$$

3. We say that a group G has property $\mathcal{P}$ if for every abelian group H, every subgroup H' of H and every morphism $f_0 : H' \to G$, there exists a morphism $f : H \to G$ such that $f = f_0$ on H'. Prove that:
a) $(\mathbf{Q}^*, \cdot)$ has not property $\mathcal{P}$; b) $(\mathbf{Q}, +)$ has property $\mathcal{P}$.

4. Let A be a ring with $0 \neq 1$. Prove that the following are equivalent:
a) A is not a field;
b) for every $n \in \mathbf{N}^*$, the equation $x^n + y^n = z^n$ has solutions in A^*.