ROMANIAN MATHEMATICAL OLYMPIAD
FINAL ROUND - 1999

IX$^{\text{th}}$ FORM

1. Let AD be the bisector of the angle A of the triangle ABC. Let M, N be on the half-lines $(AB$, respective $(AC$ such that $\sphericalangle MDA = \sphericalangle B$ and $\sphericalangle NDA = \sphericalangle C$. If $\{P\} = AD \cap MN$, prove that $AD^3 = AB \cdot AC \cdot AP$.

2. For $a, b > 0$, denote by $t(a, b)$ the positive root of the equation
$$(a + b)x^2 - 2(ab - a)x - (a + b) = 0.$$
Let $M = \{(a, b) \,|\, a \neq b \text{ and } t(a, b) \leq \sqrt{ab}\}$. Find $\min\limits_{(a,b)\in M} t(a, b)$.

3. The bisectors of the angles A and C of the convex quadrilateral $ABCD$ intersect in I. Prove that $ABCD$ is circumscriptible iff
$$[AIB] + [CID] = [AID] + [BIC].$$
($[XYZ]$ denotes area of the triangle XYZ).

4. a) Let $a, b \in \mathbf{R},\ a < b$. Prove that $x \in (a, b)$ iff exists $\lambda \in (0, 1)$ such that $x = \lambda a + (1 - \lambda)b$.
b) If the function $f : \mathbf{R} \to \mathbf{R}$ satisfy:
$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y),$$
for all $x \neq y$ and $\lambda \in (0, 1)$, then there are no parallelograms with the vertices on the graph of f.

X$^{\text{th}}$ FORM

1. Solve in $\mathbf{R}$ the system: $\begin{cases} 4^{-x} + 27^{-y} & = & \frac{5}{6} \\ \log_{27} y - \log_4 x & \geq & \frac{1}{6} \\ 27^y - 4^x & \leq & 1 \end{cases}$.

2. Let M, N, P, Q be on the edges AB, BC, CD, DA of a regular thetraedron $ABCD$. Prove that $MN \cdot NP \cdot PQ \cdot QM \geq AM \cdot BN \cdot CP \cdot DQ$.

3. Let $a, b, c \in \mathbf{C}$ such that the roots of the equation $az^2 + bz + c = 0$ are less than 1 in module. Prove that the roots w_1, w_2 of the equation $(a + \overline{c})z^2 + (b + \overline{b})z + (\overline{a} + c) = 0$ satisfy: $|w_1| = |w_2| = 1$.

4. A) Real numbers $x_1, ..., x_n, y_1, ..., y_n$ satisfy:
i) $0 < x_1 y_1 < ... < x_n y_n$; ii) $x_1 + ... + x_k \geq y_1 + ... + y_k,\ \forall\, 1 \leq k \leq n$.
Prove that $\frac{1}{x_1} + ... + \frac{1}{x_n} \leq \frac{1}{y_1} + ... + \frac{1}{y_n}$.
B) Let $A = \{a_1, ..., a_n\} \subset \mathbf{N}^*$ be such that for all $B, C \subset A$ with $B \neq C$ we have: $\sum\limits_{x\in B} x \neq \sum\limits_{x\in C} x$. Prove that $\frac{1}{a_1} + ... + \frac{1}{a_n} < 2$.

XI$^{\text{th}}$ FORM

1. Let $A \in \mathcal{M}_2(\mathbf{C})$ and define $\mathcal{C}(A) = \{B \in \mathcal{M}_2(\mathbf{C}) \,|\, AB = BA\}$. Prove that the following assertions are equivalent:
   a) $|\det(A+B)| \geq |\det B|$; b) $A^2 = 0_2$.

2. Let $z_1, ..., z_k \in \mathbf{C}$ be any two different and let $u_1, ..., u_k \in \mathbf{C}$ be such that the set
$$\{a_n = u_1 z_1^n + ... + u_k z_k^n \mid n \in \mathbf{N}^*\}$$
is finite. Prove that the sequence $(a_n)_n$ is periodic.

3. Let $f, g : [a, b] \to \mathbf{R}$ be derivable such that f' and g' are increasing and positive. Prove that there is $c \in (a, b)$ such that
$$\frac{f(b) - f(a)}{b - a} \cdot \frac{g(b) - g(a)}{b - a} = f'(c)g'(c).$$

4. Let $f : \mathbf{R} \to \mathbf{R}$ be derivable such that $f(x) = f\left(\frac{x}{2}\right) + \frac{x}{2} f'(x)$. Prove that $f(x) = ax + b$, for some a, b.

XII$^{\text{th}}$ FORM

1. Find all continuous functions $f : \mathbf{R} \to [1, \infty)$ such that there exist $a \in \mathbf{R}$ and $k \in \mathbf{N}$ satisfying: $f(x)f(2x)...f(nx) \leq an^k$, $\forall\, x \in \mathbf{R}$, $n \in \mathbf{N}^*$.

2. For any finite group G denote by $n(G)$ and $s(G)$ the number of the elements of G, respective the number of its subgroups. Are true the following assertions:
   a) for every $a > 0$, there exists a group G such that $\frac{n(G)}{s(G)} < a$;
   b) for every $a > 0$, there exists a group G such that $\frac{n(G)}{s(G)} > a$?

3. Let $a, b, c, d \in \mathbf{R}$, $ac \neq 0$ and let $f : \mathbf{R} \to \mathbf{R}$ be monotone such that
$$\int_x^{x+\sqrt{3}} f(t)dt = ax + b \quad \text{and} \quad \int_x^{x+\sqrt{2}} f(t)dt = cx + d \ , \quad \forall\, x \in \mathbf{R}.$$
Prove that $f(x) = mx + n$, for some m, n.

4. Let A be a commutative ring without zero divisors. Denote by $A[X]$ the corresponding ring of polynomials. For $n \geq 2$ define $\phi_n : A[X] \to A[X]$, $\phi_n(f) = f^n$. Prove that if the set $\{n \,|\, \phi_n \text{ is endomorphism}\}$ is non-empty, then there exists p prime such that $M = \{p, p^2, p^3, ..., p^k, ...\}$.