ROMANIAN MATHEMATICAL OLYMPIAD
FINAL ROUND - 1998

IX$^{\text{th}}$ FORM

1. Let $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbf{Z}$. Find a, b, c if $f(f(1)) = f(f(2)) = f(f(3))$.

2. If $ABCD$ is an inscriptible quadrilateral, then $|AC - BD| \leq |AB - CD|$.

3. Find $a, b \in \mathbf{Z}$ and $x \in \mathbf{Q}$ such that $abx^2 + (a^2 + b^2)x + 1 = 0$.

4. Let $A_1A_2...A_n$, $n \geq 5$ be a regular n-gon, $T \in A_1A_2 \cap A_{n-1}A_n$, $M \in \text{Int}$ (A_1A_nT). Prove that M lies on the circumcircle of the given n-gon iff

$$\sum_{i=1}^{n-1} \frac{\sin^2 \sphericalangle A_iMA_{i+1}}{d(M, A_iA_{i+1})} = \frac{\sin^2 \sphericalangle A_1MA_n}{d(M, A_1A_n)}.$$

X$^{\text{th}}$ FORM

1. Let $M = \{1, 2, ..., n\}$, $n \geq 2$ and for $1 \leq k \leq n - 1$ define

$$x_k = \frac{1}{n+1} \sum_{A \subset M,\ |A| = k} (\min A + \max A).$$

   Prove that $x_k \in \mathbf{N}$, $\forall\ 1 \leq k \leq n - 1$ and there exists i such that x_i is not divisible by 4.

2. Let $z \in \mathbf{C}$ and $a \in (1, \infty)$ be such that $|z + a| \leq a$ and $\left|z^2 + a\right| \leq a$. Prove that $|z| \leq a$.

3. Let $ABCD$ be a thetradron and $A' \in (DA)$, $B' \in (DB)$, $C' \in (DC)$, $P_c \in (AB)$, $P_a \in (BC)$, $P_b \in (AC)$, $P_c' \in (A'B')$, $P_a' \in (B'C')$ and $P_b' \in (A'C')$ be such that

$$\frac{P_cA}{P_cB} = \frac{P_c'A'}{P_c'B'} = \frac{AA'}{BB'}\ ;\ \frac{P_aB}{P_aC} = \frac{P_a'B'}{P_a'C'} = \frac{BB'}{CC'}\ ;\ \frac{P_bC}{P_bA} = \frac{P_a'C'}{P_b'A'} = \frac{CC'}{AA'}.$$

   Prove that:
   a) $AP_a \cap BP_b \cap CP_c = \{P\}$ and $A'P_a' \cap B'P_b' \cap C'P_c' = \{P'\}$.
   b) $\frac{PC}{PP_c} = \frac{P'C'}{P'P_c'}$.
   c) PP' has constant direction when A', B', C' are variable.

4. Let $n \geq 2$ and $0 < x_1 < x_2 < ... < x_n$. For $1 \leq k \leq n$ define

$$s_k = \sum_{A \subset \{x_1, ..., \mathrm{x}_n\},\ A \neq \emptyset} \frac{1}{\prod\limits_{a \in A} a}.$$

   Prove that if s_n and s_{n-1} are integer, then s_k is integer, for all $1 \leq k \leq n$.

XI$^{\text{th}}$ FORM

1. Let $A_0, A_1, ..., A_n \in \mathcal{M}_2(\mathbf{R})$, $n \geq 2$ such that $A_0 \neq aI_2$, $\forall\, a \in \mathbf{R}$ and $A_0A_k = A_kA_0$, $\forall\, 1 \leq k \leq n$. Prove that:
a) $\det\left(\sum\limits_{k=1}^{n} A_k^2\right) \geq 0$.
b) if $\det\left(\sum\limits_{k=1}^{n} A_k^2\right) = 0$ and $A_2 \neq aA_1$, $\forall\, a \in \mathbf{R}$, then $\sum\limits_{k=1}^{n} A_k^2 = 0$.

2. Let $(a_n)_{n\geq 1}$ be such that $x_n = \sum\limits_{k=1}^{n} a_k^2$ is convergent and $y_n = \sum\limits_{k=1}^{n} a_k$ is unbounded. Prove that $z_n = \{y_n\}$, $n \geq 1$ is divergent. ($\{\cdot\}$ is the fractional part function).

3. Let $f : \mathbf{R} \to \mathbf{R}$ be derivable such that $f'(x) \leq f'\left(x + \frac{1}{n}\right)$, $\forall\, x \in \mathbf{R}$ and $n \in \mathbf{N}^*$. Prove that f is continuous.

4. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function with the following property: for any $a, b \in \mathbf{R}$, $a < b$, there exist $c_1, c_2 \in [a, b]$, $c_1 \leq c_2$ such that
$$f(c_1) = \min_{x\in[a,b]} f(x) \quad \textit{and} \quad f(c_2) = \max_{x\in[a,b]} f(x).$$
Prove that f is increasing.

XII$^{\text{th}}$ FORM

1. Let $a, b, c > 0$, $a + b < 1$. Find all increasing functions $f : [0, \infty) \to [0, \infty)$ satisfying:
$$\int_0^x f(t)dt = \int_0^{ax} f(t)dt + \int_0^{bx} f(t)dt, \;\; \forall\, x \geq 0.$$

2. a) For each $p \geq 2$ denote $G_p = \bigcup\limits_{n\in\mathbf{N}} \{z \in \mathbf{C} \mid z^{p^n} = 1\}$. Prove that G_p is subgroup in $(\mathbf{C}^*, \cdot)$.
b) Let H be an infinite subgroup of $(\mathbf{C}^*, \cdot)$. Then every subgroup of H, different from H is finite iff $H = G_p$, for some p.

3. A ring A is called *Boole-ring* if $x^2 = x$, $\forall\, x \in A$. Prove that:
a) it can be defined a structure of $Boole - ring$ on a set with n elements iff $n = 2^k$, for some integer $k \geq 0$.
b) it can be defined a structure of *Boole-ring* on $\mathbf{N}$.

4. Let $K \subset \mathbf{C}$ be a field such that:
a) the field K has two endomorphisms denoted by f, g.
b) $f(x) = g(x) \Rightarrow x \in \mathbf{Q}$.
Prove that $K = \mathbf{Q}(\sqrt{d})$, for some integer d. $\in A$