ROMANIAN MATHEMATICAL OLYMPIAD
FINAL ROUND - 1996

IX$^{\text{th}}$ FORM

1. Let a, b, c be real numbers such that a and $4a+3b+2c$ have the same sign. Prove that the quadratic equation $ax^2+bx+c=0$ cannot has both roots in the interval $(1,2)$.

2. Let $a, b, c, d \in [0,1]$ and $x, y, z, t \in [0,1/2]$ be such that $a+b+c+d = x+y+z+t = 1$. Prove that:
a) $ax+by+cz+dt \geq \min\left\{\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+d}{2}, \frac{d+a}{2}, \frac{a+c}{2}, \frac{b+d}{2}\right\}$.
b) $ax+by+cz+dt \geq 54abcd$.

3. Prove that $\cos^7 x + \cos^7\left(x+\frac{2\pi}{3}\right) + \cos^7\left(x+\frac{4\pi}{3}\right) = \frac{63}{64}\cos 3x$.

4. The incircle $\mathcal{C}$ of the triangle ABC meets AB, BC, CA in D, E, F. The segments $(BE), (CF)$ meet $\mathcal{C}$ in G, H. Assuming that B and C are fixed, find the locus of the points A, D, E, F, G, H when: a) $GH \| BC$; b) $BCHG$ is inscriptible.

X$^{\text{th}}$ FORM

1. For $n, p \in \mathbf{N}$, $1 \leq p \leq n$, define $R_n^p = \sum\limits_{k=0}^{p} (p-k)^n (-1)^k \begin{pmatrix} n+1 \\ k \end{pmatrix}$. Prove that $R_n^{n-p+1} = R_n^p$.

2. Let $ABCD$ be a thetraedron and M be a variable point on the plane (BCD). The perpendiculars in M on (BCD) meets (ABC), (ACD), (ADB) in M_1, M_2, M_3. Prove that $MM_1 + MM_2 + MM_3$ is constant iff the altitude from A of the thetraedron passes through the centroid of the triangle BCD.

3. Let there be given a regular n-gon with center O and an angle $\sphericalangle XOY$ of measure $\alpha \in (0,\pi)$. Find (function of n) the values of α such that the area of the common part of the interior of the angle $\sphericalangle XOY$ and the interior of the given n-gon remains constant when $\sphericalangle XOY$ rotates around O.

4. Let a be even and b be odd. Prove that for every $n \in \mathbf{N}^*$ there exists $x \in \mathbf{N}$ such that 2^n divides ax^2+bx+c.

XI$^{\text{th}}$ FORM

1. Let I be interval and $f : I \to \mathbf{R}$ be derivable. Denote

$$J = \left\{ \frac{f(b)-f(a)}{b-a} \mid a, b \in I,\ a < b \right\}.$$

Prove that:
a) J is interval;
b) $J \subset f'(I)$ and the set $f'(I) \setminus J$ has at most 2 elements;
c) using the previous results, prove that f' has the intermediate value property.

2. a) Let $f_1, ..., f_n : \mathbf{R} \to \mathbf{R}$ be periodic and $f = f_1 + ... + f_n$. Prove that if f has finite limit to infinity, then f is constant.
b) Prove that if $a_1 \cos a_1 x + a_2 \cos a_2 x + a_3 \cos a_3 x \geq 0$, $\forall x \in \mathbf{R}$, then $a_1 a_2 a_3 = 0$.

3. Let $A, B \in \mathcal{M}_2(\mathbf{R})$ be two matrices. Prove that if $\det(AB + BA) \leq 0$, then $\det(A^2 + B^2) \geq 0$.

4. Let $A, B, C, D \in \mathcal{M}_n(\mathbf{C})$ such that A, C are invertible and $A^k B = C^k D$, $\forall k \in \mathbf{N}^*$. Prove that $B = D$.

XII$^{\text{th}}$ FORM

1. Let G be a group such that there exist exactly two elements $x, y \in G \setminus \{e\}$ with $xy = yx$. Prove that G is isomorphic with $\mathbf{Z}_3$ or with S_3. (S_3 denotes the group of permutations of 3 elements).

2. Let $f : [a, b] \to \mathbf{R}$ be monotone such that for every $x_1, x_2 \in [a, b]$, $x_1 < x_2$, there is $c \in (a, b)$ satisfying: $\int\limits_{x_1}^{x_2} f(t)dt = f(c)(x_1 - x_2)$. Prove that f is continuous on (a, b). Remains the problem true if change *monotone* with *integrable*?

3. Let A be a commutative ring such that for all $x \in A \setminus \{0\}$, there exist $m, n \in \mathbf{N}^*$ satisfying: $(x^m + 1)^n = x$. Prove that every endomorphism of A is automorphism.

4. Let $f : (0, 1] \to \mathbf{R}$ be monotone. Prove that the limits:

$$l_1 = \lim_{x \nearrow 1} \int_0^x f(t)dt \quad \text{and} \quad l_2 = \lim_{n \to \infty} \frac{1}{n} \left[f(0) + f\left(\frac{1}{n}\right) + ... + f\left(\frac{n-1}{n}\right) \right]$$

exist and $l_1 = l_2$.