## ROMANIAN MATHEMATICAL OLYMPIAD
## FINAL ROUND - 1995

### IX$^{\text{th}}$ FORM

1. Let $f : \mathbf{N}^* \to \mathbf{N}^*$ be given by $f(n) =$ number of all squares from the interval $[n^2, 2n^2]$. Prove that $f$ is increasing and surjective.

2. Let there be given an angle $\sphericalangle XOY = \alpha \in (0, \pi/2)$. Consider a variable point $M \in (OY$, $N \in (OX$ such that $MN \perp OX$, $P \in (OY$ such that $NP \perp OM$, $Q$ the midpoint of $ON$, $\{S\} = NP \cap MQ$, $T \in (OX$ such that $ST \perp OX$.
   a) Find the locus of $S$;
   b) Prove that $SP = ST$ iff $\cos 2\alpha + 2\cos\alpha = 1$.

3. Let $\Delta ABC$ and $[CX \| AB$ and $[CY$ parallel with the bisector of $\sphericalangle A$ (in the semiplane determined by $AC$ containing $B$). A variable line through $B$ meets $(CX$ in $D$ and $(CY$ in $E$. If $\{F\} = AD \cap CB$, prove that $EF$ passes through a fixed point.

4. Let $f_k : \mathbf{R} \to \mathbf{R}$, $f_k(x) = a_k x^2 + b_k x + c_k$, $k = 1, 2$, $a_1 a_2 < 0$. Prove that if the graphs of $f_1$ and $f_2$ do not intersect, then there is a line which separates them. Find such a line in case $f_1(x) = x^2 + 25/4$, $f_2(x) = -x^2 + x$.

### X$^{\text{th}}$ FORM

1. Let $a_1 < ... < a_n$. Prove that:
   a) there exist $A_1, ..., A_n \in \mathbf{R}$ such that $\prod\limits_{k=1}^{n} \dfrac{1}{x + a_k} = \sum\limits_{k=1}^{n} \frac{A_k}{x+a_k}$, $\forall\, x \neq a_k$.
   b) if $B_k = A_1 + ... + A_k$, prove that $B_k B_{k+1} < 0$, $\forall\, k = \overline{1, n}$.

2. Let $ABCDA'B'C'D'$ be a frustum of a pyramid. Find the locus of $M$ such that $vol(MADD'A') + vol(MBCC'B') = k$.

3. Let $n > 2$, $a \neq 0$ and $z \in \mathbf{C} \setminus \mathbf{R}$ be such that $z^n + az + 1 = 0$. Prove that $|z| > \sqrt[n]{\frac{1}{n-1}}$.

4. For $A = \{a_1, ..., a_p\} \subset \mathbf{R}$ denote $s_A = \sum\limits_{k=1}^{p} (-1)^{k+1} a_k$. If $M \subset \mathbf{N}$ has $n$ elements, prove that $2^{n-1}$ divides $\sum\limits_{A \subset M} s_A$.

### XI$^{\text{th}}$ FORM

1. Let $f : \mathbf{R} \to \mathbf{R}$ be surjective and if $(x_n)_{n\in\mathbf{N}}$ is divergent, then $(f(x_n))_{n\in\mathbf{N}}$ is divergent, too. Prove that $f$ is continuous.

2. Let $A_1, ..., A_k \in \mathcal{M}_n(\mathbf{R})$. Prove that $\det\left(\sum_{i=1}^{k} A_i^T \cdot A_i\right) \geq 0$.

3. Let $A \in \mathcal{M}_{3,2}(\mathbf{C})$ and $B \in \mathcal{M}_{2,3}(\mathbf{C})$ be such that $AB = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.

   Find $\det(BA)$.

4. Let $M > 0$ and $f : \mathbf{R} \to \mathbf{R}$ be continuous such that

$$|f(x+y) - f(x) - f(y)| \leq M, \ \forall\, x, y \in \mathbf{R}.$$

   Prove that exists the limit: $g(x) = \lim_{n\to\infty} \frac{f(nx)}{n}$ and $g$ is continuous in origin. Moreover, there exists, finite the limit: $\lim_{x\to\infty} \frac{f(x)}{x}$.

## XII$^{\text{th}}$ FORM

1. Let $(M, \cdot)$ be a monoid and $e$ its unity.
   a) Prove that if $M$ is finite, then there are no $a, b \in M$ such that $ab = e$ and $ba \neq e$.
   b) Prove that there exist $f, g : \mathbf{N} \to \mathbf{N}$ such that $f \circ g = \mathbf{1_N}$ and $g \circ f \neq \mathbf{1_N}$.
   c) Let $a, b \in M$ be such that $ab = e \neq ba$. If $b^n a^m = b^q a^p$, prove that $n = q$ and $m = p$.

2. Let $K$ be a finite field. Find all $n$ such that every polynomial of $n$-th degree with coefficients in $K$ does not have roots in $K$.

3. Let $f, g : [a, b] \to \mathbf{R}$ be such that:
   a) $f$ is antiderivable and is bounded from above;
   b) We have: $g(y) - g(x) \geq (y - x) \sup_{t\in[x,y]} f(t), \ \forall\, x < y$.
   Prove that $g$ is integrable.

4. Let $M$ be the set of all continuous functions $f : [0, \pi] \to \mathbf{R}$ satisfying:

$$\int_0^\pi f(x) \sin x dx = \int_0^\pi f(x) \cos x dx = 1.$$

   Find $\min_{f\in M} \int_0^\pi f^2(x) dx$.