ROMANIAN MATHEMATICAL OLYMPIAD
FINAL ROUND - 1990

X$^{\text{th}}$ FORM

1. Prove that for every positive integer $n \geq 2$ the inequality holds:

$$\frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n-1}\right)^{n}} < \left(1-\frac{1}{n^4}\right)^n.$$

2. For $n \in \mathbf{N}^*$ denote $U_n = \{x \in \mathbf{C} | x^n = 1\}$. Prove that the following are equivalent:
   (a) $\exists \alpha \in U_n$ so that $1+\alpha \in U_n$;
   (b) $\exists \beta \in U_n$ so that $1-\beta \in U_n$.

3. Let $a, b \in \mathbf{R}_+$ so that $a > b$. Let A, B, C, D be points in space so that $AB = BC = CD = a, BD = DA = AC = b$. Consider the projection E of D on the plane (ABC) and $\mathcal{C}$ the circumscribed circle of ΔABC. Prove that if $E \in \mathcal{C}$ then $\frac{a}{b} = \frac{1+\sqrt{5}}{2}$ and $D = E$.

4. (a) Prove that the number of finite sequences $a_1, a_2, ..., a_n$, where $a_i \in \{0;1\}, \sum_{i=1}^n a_i = k, 0 \leq a_i + a_{i+1} \leq 1, \forall i = 1, 2, ..., n-1$ is equal to $\binom{n-k+1}{k}$.
   (b) Let $i, n \in \mathbf{N}, i \geq 1, n \geq 2$ and a writing of n as a sum of two positive integers $n = n_1 + n_2$. Prove that:

$$\binom{n-i+1}{i} \leq \sum_{i_1, i_2 \neq 1; i_1, i_2 \geq 0} \binom{n_1 - i_1 + 1}{i_1} \cdot \binom{n_2 - i_2 + 1}{i_2} \leq \binom{n}{i}.$$