ROMANIAN MATHEMATICAL OLYMPIAD
FINAL ROUND - 1989

$\mathbf{X^{th}}$ FORM

1. Consider a tetrahedron $ABCD$ with the areas of all faces equal so that $AD = BC = a, AB = CD = b, AC = BD = c$.

   (a) In the plane (BCD) trough the vertices B, C, D consider parallel lines to the sides AD, BD, BC which determine the triangle ΔMNP. Prove that $AM \perp AN \perp AP \perp AM$.

   (b) Prove that there exists a tetrahedron with the areas of all faces equal, having sides of lengths a, b, c iff a, b, c are the lengths of a triangle with all angles less than $90°$.

2. If $a, b, c \in \mathbf{R}^*$ then the inequality holds: $\sqrt{(a-b)^2 + b^2} + \sqrt{(c-b)^2 + b^2} \geq \sqrt{a^2 + c^2}$ with equality iff $\frac{1}{|b|} = \frac{1}{|a|} + \frac{1}{|c|}$.

3. Let B, C be two points in the plane α, A in its exterior and O the projection of A on α. If $m(\angle ABC) - m(\angle ACB) \geq 90°$, prove that $m(\angle BAC) > m(\angle BOC)$.

4. (a) Determine $d = gcd(f, g)$ if $f = x^{n-1} + x^{n-2} + ... + x + 1$; $g = x^{m-1} + x^{m-2} + ... + x + 1$.

   (b) For the same d find the sum of the powers of exponent $p \in \mathbf{N}$ of the solutions of the equation $d(x) = 0$.