ROMANIAN MATHEMATICAL OLYMPIAD
FINAL ROUND - 1987

$\mathbf{X^{th}}$ FORM

1. (a) Let $p > 2$ be a prime number and $z \in \mathbf{C}, z = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}$. Determine the rational numbers $a_0, a_1, ..., a_{p-2}$ so that $\frac{1}{z-1} = a_0 + a_1 z + ... + a_{p-2}z^{p-2}$.

   (b) If $z \in \mathbf{C}, |z| = 1$ prove that $|1+z| + |1+z^2| + |1+z^3| \geq 2$.

2. Let $ABCD$ be a tetrahedron with angles less than 90° and P a point in its interior. Let P_a, P_b, P_c, P_d be the projections of P on the faces BCD, ACD, ABD, ABC of the tetrahedron. Determine the point P so that the sum $\frac{\sigma(BCD)}{PP_a} + \frac{\sigma(ACD)}{PP_b} + \frac{\sigma(ABD)}{PP_c} + \frac{\sigma(ABC)}{PP_d}$ is minimum.

3. Let $(a_n)_{n\geq 1}$ be a strictly increasing sequence of positive integers so that:

   (a) $a_{2n} = a_n + n, \forall n \geq 1$;

   (b) If a_n is prime then n is prime.

   Prove that $a_n = n, \forall n \geq 1$

4. Let p be a prime number, n a positive integer and q the quotient of the division of n to p. Prove that:
$$p \left| \left[\begin{pmatrix} n+p \\ n \end{pmatrix} + (q+1)(p-1) \right] \right. .$$