ROMANIAN MATHEMATICAL OLYMPIAD
FINAL ROUND - 1983

$\mathbf{X^{th}}$ FORM

1. Determine the function $f : \mathbf{N}^* \to (0, \infty)$, so that:
   (a) $f(4) = 4$;
   (b) $\frac{1}{f(1)f(2)} + \frac{1}{f(2)f(3)} + ... + \frac{1}{f(n)f(n+1)} = \frac{f(n)}{f(n+1)}, \forall n \in \mathbf{N}^*$.

2. Prove that $\forall k, m, n \in \mathbf{N}$ there exists $a_1, a_2, ..., a_{k-1} \in \mathbf{N}$ so that: $\left(\begin{array}{c} n+k \\ m+k-1 \end{array} \right) = \left(\begin{array}{c} n \\ m-1 \end{array} \right) + a_1 \left(\begin{array}{c} n \\ m \end{array} \right) + a_2 \left(\begin{array}{c} n \\ m+1 \end{array} \right) + ... + a_{k-1} \left(\begin{array}{c} n \\ m+k-2 \end{array} \right) + \left(\begin{array}{c} n \\ m+k-1 \end{array} \right)$.Determine the coefficients $a_1, a_2, ..., a_{k-1}$ for $k = 7$.

3. Let $z_1, z_2, z_3 \in \mathbf{C}$ so that $|z_1| = |z_2| = |z_3| = r$ and $z_2 \neq z_3$. Prove that:
$$\min_{\alpha \in \mathbf{R}} |\alpha z_1 + (1 - \alpha z_3) - z_1| = \frac{|z_1 - z_2| \cdot |z_1 - z_3|}{2r}$$

4. Let $[Ox, [Oy, [Oz$ be three semi-lines not all in the same plane and $A \in [Ox, B \in [Oy, C \in [Oz$. Prove that the following are equivalent:
   (a) The orthocenter of ΔABC is the leg of the perpendicular from O to the plane (ABC), whatever the position of A, B, C;
   (b) ΔABC has all angles less than 90°, whatever the position of A, B, C.